

The mixed Einstein-Hilbert action and extrinsic geometry of foliated manifolds

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Abstract

We develop variation formulas for the quantities of extrinsic geometry for adapted variations of metrics on almost-product (e.g. foliated) Riemannian manifolds, and apply them to study the total mixed scalar curvature of a distribution – analogue of the classical Einstein-Hilbert action. The mixed scalar curvature S_{mix} is the averaged sectional curvature over all planes that contain vectors from both distributions of an almost-product structure and the variations we consider preserve orthogonality of the distributions. We derive the directional derivative DJ_{mix} (of the total S_{mix}) for adapted variations of metrics on closed almost-product manifolds and foliations of arbitrary dimension. The obtained Euler-Lagrange equations are presented in two equivalent forms: in terms of extrinsic geometry and intrinsically using the partial Ricci tensor. Certainly, these mixed field equations admit amount of solutions (e.g., twisted products).

Keywords: Foliation; almost-product structure; mixed scalar curvature; extrinsic geometry; adapted variation; conformal; mixed Einstein-Hilbert action; twisted product

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Introduction

Foliations (i.e., decompositions of manifolds into collections of submanifolds, cf. [4]) grew out of the theory of dynamical systems; many models in mechanics and relativity are foliated (e.g., warped products). Riemannian geometry of foliations is well developed since years and has different aspects, local and global, intrinsic and extrinsic, see survey in [15, 18]. *Extrinsic geometry* of a foliation describes the properties depending on the second fundamental form of the leaves and its invariants. A Riemannian manifold may admit many kinds of geometrically interesting foliations: totally geodesic and Riemannian foliations are most popular examples.

The problem of minimizing geometric quantities has been very popular since long time: recall, for example, classical isoperimetric inequalities, Fenchel estimates of total curvature of curves and estimates of total mean curvatures of compact submanifolds, see [10]. In the context of foliations, Gluck and Ziller in 1986 considered the problem of minimizing functions like volume, total energy and bending defined for k -plane fields on Riemannian manifolds, one also has [9, 11, 17]. In all the cases mentioned above, they consider a fixed Riemannian manifold (M, g) and look for geometric objects (curves, hypersurfaces, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types.

The equations of mechanics can be also obtained as solutions of a variational problem, using a suitable functional, called the action on the configuration space. The Einstein-Hilbert action in general relativity yields the Einstein's field equations through the principle of least action. The gravitational part of the action is $J(g) = \frac{1}{k} \int_M S(g) \, d\text{vol}_g$, where g is the metric of index 1,

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$S(g)$ is the *scalar curvature* of the spacetime (M^4, g) , $k = 16\pi G/c^4$, G is the gravitational constant and c is the speed of light in vacuum. The integral is taken over the whole spacetime if it converges; otherwise, a modified definition of $J(g)$ where one integrates over arbitrarily large, relatively compact domain Ω , still yields the Einstein equations. The equations in the presence of matter are given by adding the matter action to the Hilbert-Einstein action.

On the other hand, there is definite interest on prescribing geometric quantities of given objects (see [1], and for foliations, see [5, 16, 20, 21]): given a manifold M and a geometric quantity Q (function, vector or tensor field) one may search for a Riemannian metric g on M for which a given geometric invariant (say, curvature of some sort) coincides with Q . One of the principal problems of extrinsic geometry of foliations reads as follows: *Given a foliated manifold (M, \mathcal{F}) and an extrinsic geometric property (P) of a submanifold, find a Riemannian metric g on M such that the leaves of \mathcal{F} enjoy (P) with respect to g .* For example, there exist metrics making a Reeb foliation (on the three-dimensional sphere) totally umbilical but there is no metric making a Reeb foliation harmonic (i.e., the leaves become minimal submanifolds).

In [18] the new approach from the two we just described is presented to problems in extrinsic geometry of codimension one foliations: given a foliated manifold (M, \mathcal{F}) and a property Q of a submanifold, depending on the principal curvatures of the leaves, study Riemannian metrics, which minimize the integral of Q in the class of \mathcal{F} -truncated metrics (i.e., the unit vector field N orthogonal to \mathcal{F} is the same for all metrics of the variation family). Certainly (like in some of the cases mentioned before) such Riemannian structures need not exist, but if they do, they usually have interesting geometric properties.

Let $\text{Sym}^2(M)$ be the space of all symmetric $(0, 2)$ -tensors fields tangent to M . A semi-Riemannian (pseudo-Riemannian) metric of index q on M is an element $g \in \text{Sym}^2(M)$ such that each g_x is a non-degenerate bilinear form of index q on the tangent space $T_x M$ for all $x \in M$. When $q = 0$, that is g_x is positive definite for all $x \in M$, we say that g is a Riemannian metric (resp. Lorentz metric, when $q = 1$). If a distribution $\tilde{\mathcal{D}}$ (i.e., a vector subbundle of the tangent bundle TM) is given, then a complementary distribution \mathcal{D} to $\tilde{\mathcal{D}}$ in TM can be obtained. Indeed, since M is paracompact and of class C^∞ , there exists a semi-Riemannian metric g of class C^∞ . Then we can take \mathcal{D} as the orthogonal distribution to $\tilde{\mathcal{D}}$ with respect to this metric. If $\tilde{\mathcal{D}}$ is nondegenerate (i.e., $\tilde{\mathcal{D}}_x$ is a nondegenerate subspace of the semi-Euclidean space $(T_x M, g_x)$ for every $x \in M$), then \mathcal{D} is also nondegenerate. Thus, we are entitled to consider a connected manifold M^{n+p} with a semi-Riemannian metric g and a pair of complementary orthogonal nondegenerate distributions $\tilde{\mathcal{D}}$ and \mathcal{D} of ranks $\dim_{\mathbb{R}} \tilde{\mathcal{D}}_x = n$ and $\dim_{\mathbb{R}} \mathcal{D}_x = p$ for every $x \in M$; this is called an almost-product structure on M . A. Gray and B. O'Neill calculated the curvatures of the distributions from the curvature of M , using configuration tensors (these are analogs of the second fundamental form of a submanifold). Extending the definition, we shall say that *extrinsic geometry* of an almost-product structure describes the properties, which can be expressed using configuration tensors (i.e., the integrability tensors and the second fundamental forms). In [3], a tensor calculus, adapted to decomposition

$$TM = \tilde{\mathcal{D}} \oplus \mathcal{D}, \quad (1)$$

is developed to study the geometry of both the distributions and the ambient manifolds. The sectional curvature $K(X, Y)$, where $X \in \mathcal{D}_{\mathcal{F}}$ and $Y \in \mathcal{D}$, is called *mixed*. The “mixed components” of the curvature tensor (involved in the Jacobi equation) regulate the deviation of leaves along the leaf geodesics. The *mixed scalar curvature* S_{mix} is the averaged mixed sectional curvature. In relativity, it measures the relative acceleration of two particles moving forward on neighboring geodesics.

Our main objective is to develop variation formulas for the quantities of extrinsic geometry for *adapted variations* of metrics on almost-product (e.g. foliated) Riemannian manifolds, and to apply them to study Riemannian structures on a closed manifold M , minimizing integral of the quantity $Q = S_{\text{mix}}$ for adapted (i.e., preserving the decomposition (1)) variations of metrics g_t ($g_0 = g$, $|t| < \varepsilon$), to deduce the Euler-Lagrange equations and characterize the

critical metrics in certain distinguished classes of almost-product structures. The cases of an open manifold M and general variations of metrics will be studied in further works.

This functional, called the *mixed Einstein-Hilbert action*, is imitative of Einstein-Hilbert one, except that the scalar curvature is replaced by the mixed scalar curvature. Note that the quantity $Q = S_{\text{mix}}$ is built of the invariants of extrinsic geometry of both distributions, see [24]; hence, our study belongs to the extrinsic geometry of foliations. The mixed Einstein-Hilbert action for a globally hyperbolic spacetime (M^4, g) equipped with a Bernal-Sánchez foliation was considered in [2], where Euler-Lagrange equations (called the mixed gravitational field equations) were deduced using variation formulas for the curvature and Ricci tensor, then their linearization and solution for empty space were derived.

Our approach is based on the variation formulas for the extrinsic geometry of foliations of arbitrary (co)dimension; thus, the paper shows some progress in methods of [18], where variation formulas and functionals were studied for codimension one foliations. As we shall see shortly, the Euler-Lagrange equations for the mixed Einstein-Hilbert action involve several new tensors and a new type of Ricci curvature (introduced in [16]), whose properties need to be further investigated. The questions for further study are: stability conditions, and extrinsic geometry of critical metrics with respect to adapted (and general) variations, etc.

The paper contains introduction and two chapters.

Section 1 develops variation formulas for the quantities of extrinsic geometry for adapted variations of metrics on Riemannian almost-product manifolds, and applies them to study the mixed Einstein-Hilbert action on such manifolds. Section 2 contains applications of the above to foliated manifolds (i.e., \mathcal{D} is the normal subbundle to a foliation \mathcal{F}). The main goals are the Euler-Lagrange equations for almost-product structure (Proposition 1.7) and for foliations (Theorem 2.2 and corollaries). We look at several classes of foliations: characterize critical metrics and give examples. Throughout the paper everything (manifolds, distributions, foliations, etc.) is assumed to be smooth (i.e., C^∞ -differentiable) and oriented.

1 Mixed Einstein-Hilbert action on almost-product manifolds

Using the natural representation of $O(p) \times O(n)$ on TM , A.M. Naveira obtained (cf. [12]) thirty-six distinguished classes of Riemannian almost-product manifolds $(M, g, \tilde{\mathcal{D}}, \mathcal{D})$; half of them are foliated. Following this line of research, several geometers completed the geometric interpretation, gave examples for each class, and studied the different classes of almost-product structures. The following convention is adopted for the range of indices

$$a, b, \dots \in \{1, \dots, n\}, \quad i, j, \dots \in \{1, \dots, p\}.$$

1.1 The total mixed scalar curvature and the space of adapted metrics

The *mixed scalar curvature* function on M is defined as (cf. [15, 24])

$$S_{\text{mix}} = \sum_{a,i} K(E_a, \mathcal{E}_i) = \sum_{a,i} g(R^\nabla(E_a, \mathcal{E}_i)E_a, \mathcal{E}_i), \quad (2)$$

where $\{E_a, \mathcal{E}_i\}$ is a local g -orthonormal frame on TM such that $\{E_a\}$ are tangent to $\tilde{\mathcal{D}}$. Here, $R^\nabla(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X, Y]}$ is the curvature tensor of the Levi-Civita connection $\nabla : TM \times C^\infty(TM) \rightarrow C^\infty(TM)$. The central object in the paper is the total mixed scalar curvature, referred to as the *mixed Einstein-Hilbert action* and given by

$$J_{\text{mix}}(g) = \int_M S_{\text{mix}}(g) \, d \, \text{vol}_g. \quad (3)$$

If the integral (3) does not converge, one may use the modified definition, where the region of integration is a “large” relatively compact domain $\Omega \subset M$. Note that S_{mix} is the Gaussian

curvature for surfaces foliated by curves, and $S_{\text{mix}} = \text{Ric}(N, N)$ for a codimension-one foliation with a unit normal N . In the last case, action (3) is given by (see also Sect. 2.3)

$$J_{\text{mix}}(g) = \int_M \text{Ric}_g(N, N) \, d \, \text{vol}_g, \quad (4)$$

where $\text{Ric}_g(X, Y) = \text{Tr } g(R^\nabla(X, \cdot)Y, \cdot)$ is the Ricci tensor of (M, g) .

Our considerations through the present paper are confined to the Riemannian case ($g \in \text{Riem}(M)$) and we relegate the arbitrary (and Lorentzian) signature case to further work.

Let \mathfrak{X}_M be the module over $C^\infty(M)$ of all vector fields on M (i.e., sections of the tangent bundle TM), and $\mathfrak{X}_{\mathcal{D}}$ (resp. $\mathfrak{X}_{\tilde{\mathcal{D}}}$) the module over $C^\infty(M)$ of all vector fields in \mathcal{D} (resp. $\tilde{\mathcal{D}}$). For every $X \in \mathfrak{X}_M$, let $\tilde{X} \equiv X^\top$ be the component of X along $\tilde{\mathcal{D}}$ (resp. X^\perp the component of X along \mathcal{D}) with respect to the direct sum decomposition (1). A tensor field $S \in \text{Sym}^2(M)$ is said to be *adapted* if $S(\tilde{X}, Y^\perp) = 0$ for any $X, Y \in \mathfrak{X}_M$. Let $\mathfrak{M} \equiv \mathfrak{M}(\tilde{\mathcal{D}}, \mathcal{D})$ consist of all adapted symmetric tensor fields on $(M, \tilde{\mathcal{D}}, \mathcal{D})$. The domain of J_{mix} is *a priori* the space $\text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D}) \equiv \text{Riem}(M) \cap \mathfrak{M}$ of all adapted metrics, i.e., $J_{\text{mix}} : \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D}) \rightarrow \mathbb{R}$.

We say that a tensor $S \in \text{Sym}^2(M)$ is $\tilde{\mathcal{D}}$ -truncated if $X^\perp \lrcorner S = 0$ (resp. \mathcal{D} -truncated if $\tilde{X} \lrcorner S = 0$) for any $X \in \mathfrak{X}_M$. This notion is extended for $(1, 1)$ -tensors. Let $\mathfrak{M}_{\tilde{\mathcal{D}}}$ and $\mathfrak{M}_{\mathcal{D}}$ be, respectively, the spaces of $\tilde{\mathcal{D}}$ -truncated and \mathcal{D} -truncated symmetric $(0, 2)$ -tensor fields. Then $\mathfrak{M}_{\mathcal{D}}$ and $\mathfrak{M}_{\tilde{\mathcal{D}}}$ are subspaces of \mathfrak{M} and

$$\mathfrak{M} = \mathfrak{M}_{\tilde{\mathcal{D}}} \oplus \mathfrak{M}_{\mathcal{D}}, \quad (5)$$

the decomposition is orthogonal with respect to the inner product g^* induced on \mathfrak{M} by each $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$. A tensor $S \in \mathfrak{M}_{\mathcal{D}}$ is \mathcal{D} -conformal if $S = s g^\perp$ for some $s \in C^\infty(M, \mathbb{R})$, in particular, \mathcal{D} -scaling if s is constant. Given $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$, the subspace of \mathfrak{M} , consisting of biconformal adapted tensors, splits into direct sum of \mathcal{D} - and $\tilde{\mathcal{D}}$ -conformal components, i.e., $\mathfrak{B} = \mathfrak{B}_{\tilde{\mathcal{D}}} \oplus \mathfrak{B}_{\mathcal{D}}$. Certainly, the space \mathfrak{B} is wider than the space of all conformal to g metrics.

For each $(0, 2)$ -tensor field S tangent to M we define its truncated components $\tilde{S}, S^\perp \in \Gamma(T^*M \otimes T^*M)$ by setting $\tilde{S}(X, Y) = S(\tilde{X}, \tilde{Y})$ and $S^\perp(X, Y) = S(X^\perp, Y^\perp)$ for any $X, Y \in \mathfrak{X}_M$. If $S \in \text{Sym}^2(M)$ then $S \in \mathfrak{M} \iff S = S^\perp + \tilde{S}$, see (5). In particular, for any adapted metric $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D}) \equiv \text{Riem}(M) \cap \mathfrak{M}$ one has a decomposition

$$g = g^\perp + \tilde{g} \iff g = \begin{pmatrix} g|_{\mathcal{D}}^\perp & 0 \\ 0 & \tilde{g}|_{\tilde{\mathcal{D}}} \end{pmatrix}.$$

Our purpose in this paper is to compute the directional derivatives

$$D_g J_{\text{mix}} : T_g \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D}) \equiv \mathfrak{M} \rightarrow \mathbb{R} \quad (6)$$

for any $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ on almost-product or foliated manifolds $(M, \mathcal{D}, \tilde{\mathcal{D}})$ and study the extrinsic geometry of $\tilde{\mathcal{D}}$ and \mathcal{D} in (M, g) , where g is a critical point of J_{mix} with respect to adapted variations (21). Certainly, we can restrict ourselves to the cases $D_g J_{\text{mix}} : \mathfrak{M}_{\mathcal{D}} \rightarrow \mathbb{R}$ or $D_g J_{\text{mix}} : \mathfrak{M}_{\tilde{\mathcal{D}}} \rightarrow \mathbb{R}$, when g is either a \mathcal{D} -critical point (i.e., $D_g J_{\text{mix}}(S) = 0$ for every $S \in \mathfrak{M}_{\mathcal{D}}$) or a $\tilde{\mathcal{D}}$ -critical point (i.e., $D_g J_{\text{mix}}(S) = 0$ for every $S \in \mathfrak{M}_{\tilde{\mathcal{D}}}$) of J_{mix} .

The so called musical isomorphisms \sharp and \flat will be used for arbitrary (k, l) -tensor fields, which form the infinite-dimensional vector spaces $T_l^k M$ over \mathbb{R} and modules over $C^\infty(M)$. For example, if $\omega \in T_0^1 M$ is 1-form and $X \in \mathfrak{X}_M$ then $\omega(Y) = g(\omega^\sharp, Y)$ and $X^\flat(Y) = g(X, Y)$ for any $Y \in \mathfrak{X}_M$, and if $S \in \text{Sym}^2(M)$ then the $(1, 1)$ -tensor field $S^\sharp \in T_1^1 M$ is $g(S^\sharp X, Y) = S(X, Y)$ for any $X, Y \in \mathfrak{X}_M$. Note that if $S \in \mathfrak{M}$ then $\tilde{\mathcal{D}}$ and \mathcal{D} are S^\sharp -invariant.

1.2 The fundamental tensors of almost-product manifolds

As we shall see shortly, the Euler-Lagrange equations of the variational principle associated to (6) involve a new kind of Ricci curvature (previously introduced in [16], and studied in [2] for global hyperbolic spacetimes (M^4, g)), whose properties need to be further investigated. The \mathcal{D} -truncated symmetric $(0, 2)$ -tensor field $r_{\mathcal{D}}$ on (M, g) given by

$$r_{\mathcal{D}}(X, Y) = \sum_a g(R^\nabla(E_a, X^\perp)E_a, Y^\perp), \quad X, Y \in \mathfrak{X}_M, \quad (7)$$

is referred to as the *partial Ricci tensor* concentrated on \mathcal{D} . A 2-dimensional subspace $\sigma \subset T_x M$ is *mixed* if it admits a linear basis $\{v, w\} \subset \sigma$ such that $v \in \tilde{\mathcal{D}}_x$ and $w \in \mathcal{D}_x$. Therefore, the *partial Ricci* curvature in the direction of a unit vector $X \in \mathcal{D}$ is the mean value of sectional curvatures over all mixed planes containing X . Similarly, the partial Ricci tensor on $\tilde{\mathcal{D}}$ is

$$r_{\tilde{\mathcal{D}}}(X, Y) = \sum_i g(R^\nabla(\mathcal{E}_i, \tilde{X})\mathcal{E}_i, \tilde{Y}), \quad X, Y \in \mathfrak{X}_M. \quad (8)$$

In particular (by (2)), $\text{Tr}_g r_{\tilde{\mathcal{D}}} = S_{\text{mix}} = \text{Tr}_g r_{\mathcal{D}}$. To study the partial Ricci curvature (e.g., in Proposition 1.1 below) we introduce several tensors. Let $T, h : \tilde{\mathcal{D}} \times \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and $\tilde{T}, \tilde{h} : \mathcal{D} \times \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ be the integrability tensors and the second fundamental forms of $\tilde{\mathcal{D}}$ and \mathcal{D} , respectively,

$$\begin{aligned} T(X, Y) &= (1/2) [X, Y]^\perp, & h(X, Y) &= (1/2) (\nabla_X Y + \nabla_Y X)^\perp, \\ \tilde{T}(X, Y) &= (1/2) [X, Y]^\top, & \tilde{h}(X, Y) &= (1/2) (\nabla_X Y + \nabla_Y X)^\top. \end{aligned}$$

The mean curvature vector fields of $\tilde{\mathcal{D}}$ and \mathcal{D} are $H = \text{Tr}_g h$ and $\tilde{H} = \text{Tr}_g \tilde{h}$, respectively. The distribution $\tilde{\mathcal{D}}$ (and similarly for \mathcal{D}) is called *totally umbilical*, *harmonic*, or *totally geodesic*, if $h = \frac{1}{n} H \tilde{g}$, $H = 0$, or $h = 0$, respectively.

The *conullity tensors* $\tilde{C} : \mathfrak{X}_{\tilde{\mathcal{D}}} \times \mathfrak{X}_{\mathcal{D}} \rightarrow \mathfrak{X}_{\mathcal{D}}$ and $C : \mathfrak{X}_{\mathcal{D}} \times \mathfrak{X}_{\tilde{\mathcal{D}}} \rightarrow \mathfrak{X}_{\tilde{\mathcal{D}}}$ are defined by

$$\tilde{C}_Y(X) = -(\nabla_X Y)^\perp, \quad C_Z(W) = -(\nabla_W Z)^\top, \quad (9)$$

for any $Y, W \in \mathfrak{X}_{\tilde{\mathcal{D}}}$ and $X, Z \in \mathfrak{X}_{\mathcal{D}}$. Let \tilde{A}_Y be the Weingarten operator of \mathcal{D} with respect to Y (i.e., dual to \tilde{h}), and similarly A_Z . The dual to T and \tilde{T} operators T_Z^\sharp and \tilde{T}_Y^\sharp are given by $g(T_Z^\sharp(X), Y) = g(T(X, Y), Z)$ and $g(\tilde{T}_Y^\sharp(X), Z) = g(\tilde{T}(X, Z), Y)$. Then

$$\begin{cases} \tilde{C}_Y(X) = \tilde{A}_Y(X) + \tilde{T}_Y^\sharp(X), & X \in \mathfrak{X}_{\mathcal{D}}, \\ C_Z(W) = A_Z(W) + T_Z^\sharp(W), & W \in \mathfrak{X}_{\tilde{\mathcal{D}}}. \end{cases} \quad (10)$$

The *divergence* of a vector field $\xi \in \mathfrak{X}_M$ is given by

$$\text{div } \xi = \text{Tr}(\nabla \xi) = \sum_a g(\nabla_a \xi, E_a) + \sum_i g(\nabla_i \xi, \mathcal{E}_i).$$

The Divergence Theorem is $\int_M (\text{div } \xi) d \text{vol}_g = 0$, when M is closed; this is also if M is open and ξ is supported in a relatively compact domain $\Omega \subset M$. The $\tilde{\mathcal{D}}$ -divergence of ξ is defined by $\widetilde{\text{div}} \xi = \sum_a g(\nabla_a \xi, E_a)$. Respectively, $\Delta f = \text{div}(\nabla f)$ is the Laplacian and $\tilde{\Delta} f = \widetilde{\text{div}}(\tilde{\nabla} f)$ the $\tilde{\mathcal{D}}$ -Laplacian on functions. For $X \in \mathfrak{X}_{\tilde{\mathcal{D}}}$ and a function $f \in C^2(M)$, we have, see [19],

$$\widetilde{\text{div}} X = \text{div } X + g(\tilde{H}, X), \quad \tilde{\Delta} f = \text{div}(\tilde{\nabla} f + f \tilde{H}) - (\text{div } \tilde{H})f. \quad (11)$$

Indeed, using $\tilde{H} = \sum_{i \leq p} \tilde{h}(\mathcal{E}_i, \mathcal{E}_i)$ and $g(X, \mathcal{E}_i) = 0$, one derives $(11)_1$

$$\text{div } X - \widetilde{\text{div}} X = \sum_i g(\nabla_{\mathcal{E}_i} X, \mathcal{E}_i) = - \sum_i g(h(\mathcal{E}_i, \mathcal{E}_i), X) = -g(\tilde{H}, X).$$

The identity $\text{div}(\phi \xi) = \phi \text{div } \xi + \xi(\phi)$ together with $(11)_1$ for $X = \tilde{\nabla} f$ imply $(11)_2$.

Define $(1, 1)$ -tensors, called the *Casorati operators* of \mathcal{D} (and similarly for $\tilde{\mathcal{D}}$):

$$\mathcal{A} := \sum_a A_a^2, \quad \mathcal{T} := \sum_a (T_a^\sharp)^2.$$

For $\tilde{\mathcal{D}}$ -valued $(1, 2)$ -tensors P , define $(\widetilde{\operatorname{div}} P)(X, Y) = \sum_a g((\nabla_a P)(X, Y), E_a)$. We have

$$\widetilde{\operatorname{div}} P = \operatorname{div} P + \langle \tilde{H}, P \rangle, \quad (12)$$

where $\langle P, H \rangle(X, Y) := g(P(X, Y), H)$ is a $(0, 2)$ -tensor. For example, $\widetilde{\operatorname{div}} \tilde{h} = \operatorname{div} \tilde{h} + \langle \tilde{H}, \tilde{h} \rangle$. Similarly we obtain $\operatorname{div}^\perp h = \operatorname{div} h + \langle H, h \rangle$.

The *deformation tensor* $\operatorname{Def}_{\mathcal{D}} H$ of H is the symmetric part of ∇H restricted to \mathcal{D} , i.e.,

$$2 \operatorname{Def}_{\mathcal{D}} H(X, Y) = g(\nabla_X H, Y) + g(\nabla_Y H, X), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}.$$

(We call H a *\mathcal{D} -Killing vector field* if $\operatorname{Def}_{\mathcal{D}} H = 0$). It is also useful to identify the antisymmetric part of ∇H restricted to \mathcal{D} , which is regarded as a 2-form $d_{\mathcal{D}} H$,

$$2 d_{\mathcal{D}} H(X, Y) = g(\nabla_X H, Y) - g(\nabla_Y H, X), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}.$$

Using (10) and $\operatorname{Tr}_g(A_Y T_X^\sharp) = 0$, define the \mathcal{D} -truncated $(0, 2)$ -tensor Ψ by the identity

$$\Psi(X, Y) = \operatorname{Tr}_g(C_Y C_X) = \operatorname{Tr}_g(A_Y A_X + T_Y^\sharp T_X^\sharp), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}. \quad (13)$$

Similarly we define the tensor

$$\tilde{\Psi}(X, Y) = \operatorname{Tr}_g(\tilde{A}_Y \tilde{A}_X + \tilde{T}_Y^\sharp \tilde{T}_X^\sharp), \quad X, Y \in \mathfrak{X}_{\tilde{\mathcal{D}}}.$$

Proposition 1.1. *Let $g \in \operatorname{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$. Then*

$$\begin{aligned} r_{\mathcal{D}} - \operatorname{div} \tilde{h} - \langle \tilde{H}, \tilde{h} \rangle + \tilde{\mathcal{A}}^\flat + \tilde{\mathcal{T}}^\flat + \Psi - \operatorname{Def}_{\mathcal{D}} H &= 0, \\ d_{\mathcal{D}} H + \widetilde{\operatorname{div}} \tilde{T} - \sum_a (\tilde{A}_a \tilde{T}_a^\sharp + \tilde{T}_a^\sharp \tilde{A}_a)^\flat &= 0. \end{aligned} \quad (14)$$

The equations for $r_{\tilde{\mathcal{D}}}$ are dual to (14):

$$\begin{aligned} r_{\tilde{\mathcal{D}}} - \operatorname{div} h - \langle H, h \rangle + \mathcal{A}^\flat + \mathcal{T}^\flat + \tilde{\Psi} - \operatorname{Def}_{\tilde{\mathcal{D}}} \tilde{H} &= 0, \\ d_{\tilde{\mathcal{D}}} \tilde{H} + \operatorname{div}^\perp T - \sum_i (A_i T_i^\sharp + T_i^\sharp A_i)^\flat &= 0, \end{aligned} \quad (15)$$

Proof. Let $X, Y \in \mathfrak{X}_{\mathcal{D}}$ and $U, V \in \mathfrak{X}_{\tilde{\mathcal{D}}}$, then, see [15, Lemma 2.25],

$$g(R^\nabla(U, X)V, Y) = g(((\nabla_U \tilde{C})_V - \tilde{C}_V \tilde{C}_U)(X), Y) + g(((\nabla_X C)_Y - C_Y C_X)(U), V). \quad (16)$$

Assume $\nabla_X Y \in \tilde{\mathcal{D}}_x$ and $\nabla_X E_a \in \mathcal{D}_x$ at $x \in M$. Note that

$$\begin{aligned} \sum_a g((\nabla_X C)_Y(E_a), E_a) &= \sum_a \nabla_X(g(C_Y(E_a), E_a)) \\ &= \nabla_X(g(\sum_a h(E_a, E_a), Y)) = g(\nabla_X H, Y). \end{aligned}$$

Thus, tracing (16) over $\tilde{\mathcal{D}}_x$ yields

$$r_{\mathcal{D}}(X, Y) = g(\widetilde{\operatorname{div}} \tilde{C}(X), Y) - g(\sum_a \tilde{C}_a^2(X), Y) + g(\nabla_X H, Y) - \operatorname{Tr}_g(C_Y C_X), \quad (17)$$

where $\widetilde{\operatorname{div}} \tilde{C} = \sum_{a=1}^n (\nabla_a \tilde{C})_a$. Using (17), equalities $\operatorname{Tr}_g(A_Y T_X^\sharp) = 0 = \operatorname{Tr}_g(T_Y^\sharp A_X)$ and definitions above, we find (14) as the symmetric and antisymmetric parts of (17). \square

Remark 1.1. Tracing (14)₁ over $\tilde{\mathcal{D}}$ and applying the equalities

$$\begin{aligned}\mathrm{Tr}_g \Psi &= \sum_i \mathrm{Tr}_g (A_i^2 + (T_i^\sharp)^2) = \|h\|^2 - \|T\|^2, \\ \mathrm{Tr}_g \tilde{\mathcal{A}}^\flat &= \|\tilde{h}\|^2, \quad \mathrm{Tr}_g \tilde{\mathcal{T}}^\flat = -\|\tilde{T}\|^2, \\ \mathrm{Tr}_g (\mathrm{div} \tilde{h}) &= \mathrm{div} \tilde{H}, \quad \mathrm{Tr}_g (\mathrm{Def}_{\mathcal{D}} H) = \mathrm{div}^\perp H = \mathrm{div} H + \|H\|^2\end{aligned}$$

yield the remarkable formula by P. Walczak [24],

$$S_{\mathrm{mix}} = \|H\|^2 - \|h\|^2 + \|T\|^2 + \|\tilde{H}\|^2 - \|\tilde{h}\|^2 + \|\tilde{T}\|^2 + \mathrm{div}(H + \tilde{H}), \quad (18)$$

which represents S_{mix} in terms of extrinsic geometry of the distributions. The norms of tensors can be calculated using the adapted orthonormal basis as

$$\begin{aligned}\|\tilde{h}\|^2 &= \sum_{i,j} \|\tilde{h}(\mathcal{E}_i, \mathcal{E}_j)\|^2, \quad \|\tilde{T}\|^2 = \sum_{i,j} \|\tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\|^2, \\ \|h\|^2 &= \sum_{a,b} \|h(E_a, E_b)\|^2, \quad \|T\|^2 = \sum_{a,b} \|T(E_a, E_b)\|^2.\end{aligned}$$

The difference, called the *extrinsic curvature* of $\tilde{\mathcal{D}}$ (of the leaves of \mathcal{F} in integrable case),

$$R^{\mathrm{ex}}(X, Y, Z, W) = g(h(\tilde{X}, \tilde{Z}), h(\tilde{Y}, \tilde{W})) - g(h(\tilde{X}, \tilde{Y}), h(\tilde{Z}, \tilde{W}))$$

is useful in study of extrinsic geometry of foliations, see [15, 18]. The extrinsic Ricci tensor $\mathrm{Ric}^{\mathrm{ex}}$ and the extrinsic scalar curvature function S_{ex} of $\tilde{\mathcal{D}}$ (resp., a foliation \mathcal{F}) are

$$\begin{aligned}\mathrm{Ric}^{\mathrm{ex}}(X, Y) &= \sum_a (g(h(\tilde{X}, \tilde{Y}), h(E_a, E_a)) - g(h(\tilde{X}, E_a), h(\tilde{Y}, E_a))), \\ S_{\mathrm{ex}} &= \mathrm{Tr}_g \mathrm{Ric}^{\mathrm{ex}} = \sum_a \mathrm{Ric}^{\mathrm{ex}}(E_a, E_a).\end{aligned} \quad (19)$$

Tensors \tilde{R}^{ex} , $\tilde{\mathrm{Ric}}^{\mathrm{ex}}$ and \tilde{S}_{ex} are defined similarly for \mathcal{D} . Using the equalities

$$S_{\mathrm{ex}} = \|H\|^2 - \|h\|^2, \quad \tilde{S}_{\mathrm{ex}} = \|\tilde{H}\|^2 - \|\tilde{h}\|^2,$$

we rewrite (18) as

$$S_{\mathrm{mix}} = S_{\mathrm{ex}} + \tilde{S}_{\mathrm{ex}} + \|T\|^2 + \|\tilde{T}\|^2 + \mathrm{div}(H + \tilde{H}), \quad (20)$$

Note that $\|H\|^2 \leq n \|h\|^2$ with the equality for totally umbilical $\tilde{\mathcal{D}}$.

1.3 Variation formulas for the extrinsic geometry

To apply the methods of variational calculus to J_{mix} , given an adapted metric g on $(M, \tilde{\mathcal{D}}, \mathcal{D})$, we shall consider smooth 1-parameter variations of $g_0 = g$,

$$\{g_t \in \mathrm{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D}) : |t| < \varepsilon\}. \quad (21)$$

If the integral (3) does not converge then the induced infinitesimal variations $S_t \equiv (\partial g_t / \partial t) \in \mathfrak{M}$ are supported in a sufficiently large relatively compact domain $\Omega \subset M$. We adopt the notations

$$\partial_t \equiv \partial / \partial t, \quad S_t \equiv \partial_t g_t, \quad S \equiv \{\partial_t g_t\}_{t=0}. \quad (22)$$

By taking into account the decomposition (5), it will suffice to work with special curves $\{g_t\}_{|t| < \varepsilon}$ issuing at $g \in \mathrm{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ termed *$\tilde{\mathcal{D}}$ -variations* (resp. *\mathcal{D} -variations*), as the associated infinitesimal variation S lies in $\mathfrak{M}_{\mathcal{D}}$ (resp. in $\mathfrak{M}_{\tilde{\mathcal{D}}}$).

An adapted variation of a metric $\underline{g} \in \mathrm{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ has the form $\{g_t^\perp + \tilde{g}_t : |t| < \varepsilon\}$. The corresponding \mathcal{D} -variation (resp. $\tilde{\mathcal{D}}$ -variation) of g has the form

$$\{g_t = g_t^\perp + \tilde{g} : |t| < \varepsilon\}, \quad (\text{resp. } \{g_t = g^\perp + \tilde{g}_t : |t| < \varepsilon\}). \quad (23)$$

For variations (21)–(22) we have, see for example [18],

$$2g_t(\partial_t(\nabla_X^t Y), Z) = (\nabla_X^t S)(Y, Z) + (\nabla_Y^t S)(X, Z) - (\nabla_Z^t S)(X, Y), \quad (24)$$

where $X, Y, Z \in \mathfrak{X}_M$ and the first covariant derivative of a $(0, 2)$ -tensor S is expressed as

$$\nabla_Z S(Y, V) = Z(S(Y, V)) - S(\nabla_Z Y, V) - S(Y, \nabla_Z V).$$

Lemma 1.2. *Let a local $(\tilde{\mathcal{D}}, \mathcal{D})$ -adapted frame $\{E_a, \mathcal{E}_i\}$ evolve by (21)–(22) according to*

$$\partial_t E_a = -(1/2) S^\sharp(E_a), \quad \partial_t \mathcal{E}_i = -(1/2) S^\sharp(\mathcal{E}_i). \quad (25)$$

Then $\{E_a(t), \mathcal{E}_i(t)\}$ is a g_t -orthonormal frame adapted to $(\tilde{\mathcal{D}}, \mathcal{D})$ for all t .

Proof. For $\{E_a(t)\}$ (and similarly for $\{\mathcal{E}_i(t)\}$) we have

$$\begin{aligned} \partial_t(g_t(E_a, E_b)) &= g_t(\partial_t E_a(t), E_b(t)) + g_t(E_a(t), \partial_t E_b(t)) + (\partial_t g_t)(E_a(t), E_b(t)) \\ &= S(E_a(t), E_b(t)) - \frac{1}{2} g_t(S^\sharp(E_a(t)), E_b(t)) - \frac{1}{2} g_t(E_a(t), S^\sharp(E_b(t))) = 0. \quad \square \end{aligned}$$

Lemma 1.3 (Cf. [19]). *For \mathcal{D} -variations (21)–(22) we have $\partial_t T = 0$, $\partial_t \tilde{T} = 0$ and*

$$2\partial_t \tilde{h}(X, Y) = (\tilde{h} - \tilde{T})(S^\sharp(X), Y) + (\tilde{h} + \tilde{T})(X, S^\sharp(Y)) - \tilde{\nabla} S(X, Y), \quad (26)$$

$$2\partial_t \tilde{H} = -\tilde{\nabla}(\text{Tr } S^\sharp), \quad \partial_t h = -S^\sharp \circ h, \quad \partial_t H = -S^\sharp(H). \quad (27)$$

For \mathcal{D} -conformal variations, i.e., $S = s g^\perp$ where $s : M \rightarrow \mathbb{R}$ is a smooth function, we have

$$\partial_t \tilde{h} = s \tilde{h} - (1/2) (\tilde{\nabla} s) g^\perp, \quad \partial_t \tilde{H} = -(p/2) \tilde{\nabla} s, \quad (28)$$

$$\partial_t h = -sh, \quad \partial_t H = -sH. \quad (29)$$

The formulas for $\tilde{\mathcal{D}}$ -variations and $\tilde{\mathcal{D}}$ -conformal variations are dual to (26)–(27) and (28)–(29). Hence, \mathcal{D} -variations preserve total umbilicity, total geodesy and harmonicity of $\tilde{\mathcal{D}}$. Moreover, \mathcal{D} -conformal variations preserve total umbilicity of \mathcal{D} .

Proof. Recall that $\tilde{C}_N = \tilde{A}_N + \tilde{T}_N^\sharp$ is the conullity operator of \mathcal{D} relative to the unit vector $N \in \tilde{\mathcal{D}}$. Using (24), the symmetry of S , and the property $S(\cdot, \tilde{\mathcal{D}}) = 0$, we obtain

$$\begin{aligned} 2g_t(\partial_t \tilde{h}(X, Y), N) &= g_t(\partial_t(\nabla_X^t Y + \nabla_Y^t X), N) \\ &= (\nabla_X^t S)(Y, N) + (\nabla_Y^t S)(X, N) - (\nabla_N^t S)(X, Y) \\ &= S(\tilde{C}_N(X), Y) + S(\tilde{C}_N(Y), X) - (\nabla_N^t S)(X, Y) \end{aligned}$$

for all $X, Y \in \mathcal{D}$ and a unit vector $N \in \tilde{\mathcal{D}}$. The above and

$$S(\tilde{C}_N(X), Y) = g((\tilde{A}_N + \tilde{T}_N^\sharp)(X), S^\sharp(Y)) = g((\tilde{h} + \tilde{T})(X, S^\sharp(Y)), N)$$

yield (26). Next, tracing (26) and using skew-symmetry of \tilde{T} , we deduce (27)₁,

$$\begin{aligned} 2g(\partial_t \tilde{H}, X) &= 2 \sum_i g(\partial_t(\tilde{h}(\mathcal{E}_i, \mathcal{E}_i)), X) \\ &= 2 \sum_i g(\partial_t \tilde{h}(\mathcal{E}_i, \mathcal{E}_i) + 2\tilde{h}(\partial_t \mathcal{E}_i, \mathcal{E}_i), X) = - \sum_i (\nabla_X^t S)(\mathcal{E}_i, \mathcal{E}_i) = -X(\text{Tr } S^\sharp). \end{aligned}$$

By (24), for any $X \in \mathcal{D}$ and $E_a, E_b \in \mathfrak{X}_{\tilde{\mathcal{D}}}$,

$$\begin{aligned} 2g_t(\partial_t h(E_a, E_b), X) &= 2g_t(\partial_t(\nabla_a^t E_b + \nabla_b^t E_a), X) \\ &= (\nabla_a^t S)(X, E_b) + (\nabla_b^t S)(X, E_a) - (\nabla_X^t S)(E_a, E_b) \\ &= -S(\nabla_a^t E_b, X) - S(\nabla_b^t E_a, X) = -2S(h(E_a, E_b), X). \end{aligned}$$

This proves (27)₂. Next (by $S(E_a, \cdot) = 0$ and the equality $H = \sum_a h(E_a, E_a)$ for a local orthonormal frame (E_a) of $\tilde{\mathcal{D}}$) we may derive (27)₃

$$\begin{aligned} g_t(\partial_t H, X) &= \sum_a g_t(\partial_t(\nabla_a E_a), X) = (\nabla_a S)(E_a, X) - (1/2)(\nabla_X S)(E_a, E_a) \\ &= -\sum_a S(\nabla_a E_a, X) = -S(H, X) = -g(S^\sharp(H), X). \end{aligned}$$

For \mathcal{D} -conformal variations we substitute $S = s g^\perp$ into (26) and obtain (28), also (29) follows directly from (27). Recall that total umbilicity of $\tilde{\mathcal{D}}$ means $h = \frac{1}{p} H g|_{\tilde{\mathcal{D}}}$. By (27) we have

$$\partial_t(h - (1/p)H g|_{\tilde{\mathcal{D}}}) = -S^\sharp \circ (h - (1/p)H g|_{\tilde{\mathcal{D}}}).$$

Thus, the last claim in the lemma is a consequence of results about solutions to ODEs. \square

Define \mathcal{D} -truncated symmetric $(0, 2)$ -tensor Φ_h using the identity (with arbitrary S)

$$\langle \Phi_h, S \rangle = S(H, H) - \sum_{a,b} S(h(E_a, E_b), h(E_a, E_b)) \quad (30)$$

that vanishes when $n = 1$. We have $\text{Tr}_g \Phi_h = S_{\text{ex}}$ and $\text{Tr}_g \Phi_T = -\|T\|^2$. Using the skew-symmetry of T , define the \mathcal{D} -truncated symmetric $(0, 2)$ -tensor Φ_T by

$$\langle \Phi_T, S \rangle = -\sum_{a,b} S(T(E_a, E_b), T(E_a, E_b)). \quad (31)$$

Similarly, define $\tilde{\mathcal{D}}$ -truncated tensors $\Phi_{\tilde{h}}$ and $\Phi_{\tilde{T}}$, and get $\text{Tr}_g \Phi_{\tilde{h}} = \tilde{S}_{\text{ex}}$ and $\text{Tr}_g \Phi_{\tilde{T}} = -\|\tilde{T}\|^2$.

Define a self-adjoint $(1, 1)$ -tensor with zero trace (and similarly its dual tensor $\tilde{\mathcal{K}}$)

$$\tilde{\mathcal{K}} = \sum_a [\tilde{T}_a^\sharp, \tilde{A}_a] = \sum_a (\tilde{T}_a^\sharp \tilde{A}_a - \tilde{A}_a \tilde{T}_a^\sharp).$$

Observe that if \mathcal{D} is integrable then $\tilde{T}_a^\sharp = 0$ for all $a \in \{1, \dots, n\}$, hence $\tilde{\mathcal{K}} = 0$. Also, if \mathcal{D} is totally umbilical, then every operator \tilde{A}_a is a multiple of identity and $\tilde{\mathcal{K}}$ vanishes as well.

Lemma 1.4. *For \mathcal{D} -variations (21)–(22) we have*

$$\begin{aligned} \partial_t \|\tilde{h}\|^2 &= \langle \text{div } \tilde{h} - \tilde{\mathcal{K}}^\flat, S \rangle - \text{div}(\langle \tilde{h}, S \rangle), \\ \partial_t \|\tilde{H}\|^2 &= \langle (\text{div } \tilde{H}) g, S \rangle - \text{div}((\text{Tr}_g S) \tilde{H}), \\ \partial_t (\|h\|^2 - \|H\|^2) &= \langle \Phi_h, S \rangle, \\ \partial_t \|\tilde{T}\|^2 &= \langle 2 \tilde{\mathcal{T}}^\flat, S \rangle, \quad \partial_t \|T\|^2 = -\langle \Phi_T, S \rangle. \end{aligned} \quad (32)$$

For \mathcal{D} -conformal variations (21)–(22), we have

$$\begin{aligned} \partial_t \|\tilde{h}\|^2 &= s \text{div } \tilde{H} - \text{div}(s \tilde{H}), \\ \partial_t \|\tilde{H}\|^2 &= p (s \text{div } \tilde{H} - \text{div}(s \tilde{H})), \\ \partial_t \|h\|^2 &= -s \|h\|^2, \quad \partial_t \|H\|^2 = -s \|H\|^2, \\ \partial_t \|\tilde{T}\|^2 &= -2s \|\tilde{T}\|^2, \quad \partial_t \|T\|^2 = s \|T\|^2. \end{aligned} \quad (33)$$

The formulas for $\tilde{\mathcal{D}}$ -variations are dual to (32) and (33).

Proof. Assume $\nabla_a \mathcal{E}_i \in \mathcal{D}_x$ at a point $x \in M$. We calculate using (24) and Lemmas 1.2 and 1.3. First we obtain (32)₁:

$$\begin{aligned} \partial_t \|\tilde{T}\|^2 &= 2 \sum_{i,j,a} g(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a) g(\tilde{T}(\partial_t \mathcal{E}_i, \mathcal{E}_j) + \tilde{T}(\mathcal{E}_i, \partial_t \mathcal{E}_j), E_a) \\ &= -\sum_{i,j,a} g(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a) g(\tilde{T}(S^\sharp(\mathcal{E}_i), \mathcal{E}_j) + \tilde{T}(\mathcal{E}_i, S^\sharp(\mathcal{E}_j)), E_a) \\ &= -\sum_{i,j,a} g(\tilde{T}_a^\sharp(\mathcal{E}_i), \mathcal{E}_j) g((\tilde{T}_a^\sharp S^\sharp + S^\sharp \tilde{T}_a^\sharp)(\mathcal{E}_i), \mathcal{E}_j) \\ &= -\sum_{i,a} g((\tilde{T}_a^\sharp S^\sharp + S^\sharp \tilde{T}_a^\sharp)(\mathcal{E}_i), \tilde{T}_a^\sharp(\mathcal{E}_i)) = \sum_{i,a} g(((\tilde{T}_a^\sharp)^2 S^\sharp + \tilde{T}_a^\sharp S^\sharp \tilde{T}_a^\sharp)(\mathcal{E}_i), \mathcal{E}_i) \\ &= 2 \sum_a \text{Tr}_g((\tilde{T}_a^\sharp)^2 S^\sharp) = 2 \text{Tr}_g(\tilde{\mathcal{T}} S^\sharp) = \langle 2 \tilde{\mathcal{T}}^\flat, S \rangle. \end{aligned}$$

Next, using (12), we obtain (32)₁:

$$\begin{aligned}
\partial_t \|\tilde{h}\|^2 &= 2 \sum_{i,j,a} g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a) g(\partial_t(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j)), E_a) \\
&= 2 \sum_{i,j,a} g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a) g((\partial_t \tilde{h})(\mathcal{E}_i, \mathcal{E}_j) + \tilde{h}(\partial_t \mathcal{E}_i, \mathcal{E}_j) + \tilde{h}(\mathcal{E}_i, \partial_t \mathcal{E}_j), E_a) \\
&= \sum_{i,j,a} g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a) (g(\tilde{T}(\mathcal{E}_i, S^\sharp(\mathcal{E}_j)) - \tilde{T}(S^\sharp(\mathcal{E}_i), \mathcal{E}_j), E_a) - \nabla_a S(\mathcal{E}_i, \mathcal{E}_j)) \\
&= \sum_{i,j,a} \left(g(\tilde{A}_a(\mathcal{E}_i), \mathcal{E}_j) g([S^\sharp, \tilde{T}_a^\sharp](\mathcal{E}_i), \mathcal{E}_j) - \nabla_a (g(S(\mathcal{E}_i, \mathcal{E}_j) \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a)) \right. \\
&\quad \left. - \nabla_a g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a) S(\mathcal{E}_i, \mathcal{E}_j) \right) = \langle \widetilde{\text{div}} \tilde{h} - g(\tilde{h}, \tilde{H}) + \tilde{\mathcal{K}}^\flat, S \rangle - \text{div}(\langle \tilde{h}, S \rangle).
\end{aligned}$$

Next, using $S(\tilde{H}, \tilde{H}) = 0$ (since S is \mathcal{D} -truncated) we obtain

$$\partial_t \|\tilde{H}\|^2 = \partial_t g(\tilde{H}, \tilde{H}) = 2g(\partial_t \tilde{H}, \tilde{H}) = -g(\nabla(\text{Tr } S^\sharp), \tilde{H}).$$

Note that $g(\nabla(\text{Tr } S^\sharp), \tilde{H}) = \text{div}((\text{Tr } S^\sharp)\tilde{H}) - (\text{div } \tilde{H}) \text{Tr } S^\sharp$; hence, (32)₂ follows. We have

$$\begin{aligned}
\partial_t \|H\|^2 &= \partial_t g(H, H) = S(H, H) + 2g(\partial_t H, H) \\
&= S(H, H) - 2g(S^\sharp(H), H) = -S(H, H), \\
\partial_t \|h\|^2 &= \partial_t \sum_{i,a,b} g(h(E_a, E_b), \mathcal{E}_i)^2 \\
&= 2 \sum_{i,a,b} g(h(E_a, E_b), \mathcal{E}_i) \partial_t g(h(E_a, E_b), \mathcal{E}_i) \\
&= - \sum_{i,a,b} g(h(E_a, E_b), \mathcal{E}_i) g(h(E_a, E_b), S^\sharp(\mathcal{E}_i)) \\
&= - \sum_{a,b} S(h(E_a, E_b), h(E_a, E_b)).
\end{aligned}$$

From the above, (32)₃ follows. Finally, we have (32)₄:

$$\begin{aligned}
\partial_t \|T\|^2 &= \partial_t \sum_{i,a,b} g(T(E_a, E_b), \mathcal{E}_i)^2 \\
&= 2 \sum_{i,a,b} g(T(E_a, E_b), \mathcal{E}_i) \partial_t (g(T(E_a, E_b), \mathcal{E}_i)) \\
&= 2 \sum_{i,a,b} g(T(E_a, E_b), \mathcal{E}_i) (S(T(E_a, E_b), \mathcal{E}_i) + g(T(E_a, E_b), \partial_t \mathcal{E}_i)) \\
&= \sum_{i,a,b} g(T(E_a, E_b), \mathcal{E}_i) g(T(E_a, E_b), S^\sharp(\mathcal{E}_i)) \\
&= \sum_{a,b} S(T(E_a, E_b), T(E_a, E_b)).
\end{aligned}$$

For \mathcal{D} -conformal case we use $\nabla_a S = (\nabla_a s) g^\perp$ and the above. \square

1.4 Directional derivatives of J_{mix} on almost-product manifolds

In this section we compute directional derivatives $D_g J_{\text{mix}}$ of the functional (6) on a closed Riemannian manifold (M, g) with almost-product structure. For any $f \in L^1(M, d\text{vol}_g)$, denote by

$$f(M, g) := \frac{1}{\text{Vol}(M, g)} \int_M f d\text{vol}_g$$

the mean value of f on M with respect to $d\text{vol}_g$. Together with g_t of (23), consider the metrics

$$\bar{g}_t = \phi_t g_t^\perp + \tilde{g}, \quad \phi_t \equiv (\text{Vol}(M, g_t)/\text{Vol}(M, g))^{-2/p}, \quad |t| < \varepsilon \quad (34)$$

$$\text{respectively, } \bar{g}_t = g^\perp + \phi_t \tilde{g}_t, \quad \phi_t \equiv (\text{Vol}(M, g_t)/\text{Vol}(M, g))^{-2/n}, \quad |t| < \varepsilon.$$

We will show that $\text{Vol}(M, \bar{g}_t) = \text{Vol}(M, g)$ for all t .

Proposition 1.5. *The \mathcal{D} -variations (respectively, $\tilde{\mathcal{D}}$ -variations) of the functional (3), corresponding to \bar{g}_t and g_t , are related by*

$$\frac{d}{dt}\{J_{\text{mix}}(\bar{g}_t)\}_{t=0} = \frac{d}{dt}\{J_{\text{mix}}(g_t)\}_{t=0} - \frac{1}{2}S_{\text{mix}}^*(M, g) \int_M (\text{Tr}_g S) d\text{vol}_g, \quad (35)$$

where

$$S_{\text{mix}}^*(M, g) = \begin{cases} S_{\text{mix}}(M, g) - \frac{2}{p}(S_{\text{ex}} + 2\|\tilde{T}\|^2 - \|T\|^2)(M, g) & \text{for } \mathcal{D}\text{-variations,} \\ S_{\text{mix}}(M, g) - \frac{2}{n}(\tilde{S}_{\text{ex}} - \|\tilde{T}\|^2 + 2\|T\|^2)(M, g) & \text{for } \tilde{\mathcal{D}}\text{-variations.} \end{cases} \quad (36)$$

Proof. By (20) and the Divergence Theorem, we have

$$J_{\text{mix}}(g) = \int_M Q(g) d\text{vol}_g, \quad (37)$$

where $Q(g) := S_{\text{mix}} - \text{div}(H + \tilde{H})$ is represented using (20) as

$$Q(g) = S_{\text{ex}}(g) + \tilde{S}_{\text{ex}}(g) + \|T\|_g^2 + \|\tilde{T}\|_g^2. \quad (38)$$

The volume form evolves as (cf. [18])

$$\partial_t (d\text{vol}_{g_t}) = \frac{1}{2}(\text{Tr}_{g_t} S_t) d\text{vol}_{g_t}. \quad (39)$$

Thus,

$$\frac{d}{dt}\{J_{\text{mix}}(g_t)\}_{t=0} = \int_M \left\{ \partial_t Q(g_t)|_{t=0} + \frac{1}{2}(S_{\text{mix}}(g) - \text{div}(H + \tilde{H})) \text{Tr}_g S \right\} d\text{vol}_g. \quad (40)$$

Let us fix a \mathcal{D} -variation (23)₁, the case of $\tilde{\mathcal{D}}$ -variations is studied similarly. As \bar{g}_t are \mathcal{D} -conformal to g_t with constant scale ϕ_t , their volume forms are related as

$$d\text{vol}_{\bar{g}_t} = \phi_t^{p/2} d\text{vol}_{g_t}; \quad (41)$$

hence, $\text{Vol}(M, \bar{g}_t) = \int_M d\text{vol}_{\bar{g}_t} = \text{Vol}(M, g)$. Let us differentiate in (41) so that to obtain

$$\begin{aligned} \partial_t (d\text{vol}_{\bar{g}_t}) &= (\phi_t^{p/2})' d\text{vol}_{g_t} + \phi_t^{p/2} \partial_t (d\text{vol}_{g_t}) \\ &= \frac{1}{2} \left(\text{Tr } S_t^\sharp - \frac{1}{\text{Vol}(M, g_t)} \int_M (\text{Tr}_{g_t} S_t) d\text{vol}_{g_t} \right) d\text{vol}_{\bar{g}_t}. \end{aligned}$$

Here we used (39) and the fact that $\phi_0 = 1$ and

$$\begin{aligned} \phi_t' &= -\frac{2}{p} \left(\frac{\text{Vol}(M, g_t)}{\text{Vol}(M, g)} \right)^{-\frac{2}{p}-1} \frac{1}{\text{Vol}(M, g)} \int_M \partial_t (d\text{vol}_{g_t}) \\ &= -\frac{\phi_t}{p \text{Vol}(M, g_t)} \int_M (\text{Tr}_{g_t} S_t) d\text{vol}_{g_t}. \end{aligned}$$

For the \mathcal{D} -scaling $\bar{g} = \phi g^\perp + \tilde{g}$ of $g = g^\perp + \tilde{g}$, using (33), we have

$$\begin{aligned} \|T\|_{\bar{g}}^2 &= \phi \|T\|_g^2, \quad \|\tilde{T}\|_{\bar{g}}^2 = \phi^{-2} \|\tilde{T}\|_g^2, \\ \|h\|_{\bar{g}}^2 &= \phi^{-1} \|h\|_g^2, \quad \|H\|_{\bar{g}}^2 = \phi^{-1} \|H\|_g^2, \quad \|\tilde{h}\|_{\bar{g}}^2 = \|\tilde{h}\|_g^2, \quad \|\tilde{H}\|_{\bar{g}}^2 = \|\tilde{H}\|_g^2. \end{aligned}$$

Substituting into $Q(\bar{g})$ due to definition (38), we obtain

$$Q(\bar{g}) = Q(g) + (\phi^{-1} - 1) S_{\text{ex}}(g) + (\phi^{-2} - 1) \|\tilde{T}\|_g^2 + (\phi - 1) \|T\|_g^2.$$

(For example, $Q(\bar{g}) = Q(g)$ when $n = 1$ and $\tilde{T} = 0$). The derivation of the above yields

$$\partial_t Q(\bar{g}_t)|_{t=0} = \partial_t Q(g_t)|_{t=0} - \phi'_0 (S_{\text{ex}}(g) + 2 \|\tilde{T}\|_g^2 - \|T\|_g^2),$$

where $\phi'_0 = -\frac{1}{p} \frac{1}{\text{Vol}(M, g)} \int_M (\text{Tr}_g S) d \text{vol}_g$. Hence, the \mathcal{D} -variation of the functional (3) is

$$\begin{aligned} & \frac{d}{dt} \{J_{\text{mix}}(\bar{g}_t)\}_{t=0} \\ &= \int_M \left\{ \partial_t Q(\bar{g}_t)|_{t=0} + \frac{1}{2} Q(g) \left(\text{Tr}_g S - \frac{1}{\text{Vol}(M, g)} \int_M (\text{Tr}_g S) d \text{vol}_g \right) \right\} d \text{vol}_g \\ &= \int_M \partial_t Q(g_t)|_{t=0} d \text{vol}_g + \frac{1}{2} \int_M Q(g) (\text{Tr}_g S) d \text{vol}_g + \int_M (\text{Tr}_g S) d \text{vol}_g \times \\ & \quad \times \frac{1}{2 \text{Vol}(M, g)} \int_M \left[\frac{2}{p} (S_{\text{ex}} + 2 \|\tilde{T}\|_g^2 - \|T\|_g^2) - Q(g) \right] d \text{vol}_g \\ &= \frac{d}{dt} \{J_{\text{mix}}(g_t)\}_{t=0} - \frac{1}{2} S_{\text{mix}}^*(M, g) \int_M (\text{Tr}_g S) d \text{vol}_g, \quad \square \end{aligned}$$

Remark 1.2. It should be observed that we work with two kinds of variations, (23) and (34); the last one of which preserves the volume of M . Formulas containing $S_{\text{mix}}^*(M, g)$ correspond to 1-parameter variations of the form (34). To obtain similar formulas, corresponding to 1-parameter variations of the form (23), one should merely delete the mean value terms $S_{\text{mix}}^*(M, g)$ in the previous identities.

Example 1.6 (Variations not preserving the volume of (M, g)). In general, variations g_t of (23) do not preserve the volume of (M, g) , and for critical metrics one may obtain only trivial solutions $J_{\text{mix}}(g) = 0$. Let $\tilde{\mathcal{D}}$ be a codimension one integrable distribution (i.e., $p = 1$ and $T = 0$) on a closed Riemannian manifold (M^{n+1}, g) with $g = g^\perp + \tilde{g}$, see Sect. 2.3. Define the functions $\tau_1 = \text{Tr}_g h$ and $\tau_2 = \|h\|_g^2$. For a canonical \mathcal{D} -variation $g_t = g_t^\perp + \tilde{g}$, i.e., $g_t^\perp = (1+t)g_0^\perp$ ($|t| < 1$), we have $\partial_t g_t = s g_t^\perp$ with $s = 1/(1+t)$, and $d \text{vol}_t = (1+t)^{1/2} d \text{vol}$. Since $(\tau_1^2 - \tau_2)_t = (\tau_1^2 - \tau_2)_0/(1+t)$, we find

$$J_{\text{mix}}(g_t) = \int_M (\tau_1^2 - \tau_2) d \text{vol}_t = (1+t)^{-\frac{1}{2}} J_{\text{mix}}(g) \Rightarrow \frac{d}{dt} \{J_{\text{mix}}(g_t)\}_{t=0} = -\frac{1}{2} J_{\text{mix}}(g).$$

Thus, if g is a critical point of the functional J_{mix} , then $J_{\text{mix}}(g) = 0$.

1.5 Euler-Lagrange equations for J_{mix} on almost-product manifolds

In this section we write down Euler-Lagrange equations of the variational principle $\delta J_{\text{mix}}(g) = 0$ on a closed manifold M with almost-product structure.

Proposition 1.7. *Let $\tilde{\mathcal{D}}$ and \mathcal{D} be complementary distributions on a closed manifold M . If $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ is a critical point of J_{mix} with respect to variations (21)–(22), then the following Euler-Lagrange equations (relating the quantities of extrinsic geometry) are satisfied:*

$$\begin{aligned} & \text{div}(\tilde{h} - \tilde{H} g^\perp) - 2 \tilde{\mathcal{T}}^\flat + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} + \tilde{\mathcal{K}}^\flat \\ &= \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} + \|T\|^2 + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \end{aligned} \tag{42}$$

$$\begin{aligned} & \text{div}(h - H \tilde{g}) - 2 \mathcal{T}^\flat + \Phi_h + \Phi_{\tilde{T}} + \mathcal{K}^\flat \\ &= \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} + \|T\|^2 + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) \tilde{g} \quad (\text{for } \tilde{\mathcal{D}}\text{-variations}). \end{aligned} \tag{43}$$

Using the partial Ricci tensor, we rewrite (42) and (43) as

$$\begin{aligned} r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \tilde{\mathcal{A}}^b - \tilde{\mathcal{T}}^b + \Phi_h + \Phi_T + \Psi - \text{Def}_{\mathcal{D}} H + \tilde{\mathcal{K}}^b \\ = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \text{div}(\tilde{H} - H)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \end{aligned} \quad (44)$$

$$\begin{aligned} r_{\tilde{\mathcal{D}}} - \langle h, H \rangle + \mathcal{A}^b - \mathcal{T}^b + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} + \tilde{\Psi} - \text{Def}_{\tilde{\mathcal{D}}} \tilde{H} + \mathcal{K}^b \\ = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \text{div}(H - \tilde{H})) \tilde{g} \quad (\text{for } \tilde{\mathcal{D}}\text{-variations}). \end{aligned} \quad (45)$$

Proof. Since (43) is dual to (42), we shall prove (42) only. By Lemma 1.4 to (38), and using (12) and the Divergence Theorem, we obtain

$$\int_M \partial_t Q \, d\text{vol}_g = \int_M \langle -\text{div}(\tilde{h} - \tilde{H} g^\perp) + 2\tilde{\mathcal{T}}^b - \Phi_h - \Phi_T - \tilde{\mathcal{K}}^b, S \rangle d\text{vol}_g, \quad (46)$$

where $S = \{\partial_t g_t\}_{|t=0} \in \mathfrak{M}$. Note that $\text{Tr}_g S = \langle S, g \rangle$. Then by (40), we have

$$\begin{aligned} \frac{d}{dt} \{J_{\text{mix}}(g_t)\}_{t=0} &= \int_M \langle -\text{div}(\tilde{h} - \tilde{H} g^\perp) + 2\tilde{\mathcal{T}}^b - \Phi_h - \Phi_T - \tilde{\mathcal{K}}^b \\ &\quad + \frac{1}{2} (S_{\text{mix}} - \text{div}(H + \tilde{H})) g, S \rangle d\text{vol}_g. \end{aligned} \quad (47)$$

By (47) and Proposition 1.5, we obtain

$$\begin{aligned} \frac{d}{dt} \{J_{\text{mix}}(\bar{g}_t)\}_{t=0} &= \int_M \langle -\text{div}(\tilde{h} - \tilde{H} g^\perp) + 2\tilde{\mathcal{T}}^b - \Phi_h - \Phi_T - \tilde{\mathcal{K}}^b \\ &\quad + \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) - \text{div}(\tilde{H} + H)) g, S \rangle d\text{vol}_g. \end{aligned} \quad (48)$$

If g is a critical point of J_{mix} with respect to \mathcal{D} -variations, then the integral in (48) is zero for arbitrary \mathcal{D} -truncated tensor $S \in \mathfrak{M}$, that yields

$$\text{div}(\tilde{h} - \tilde{H} g^\perp) - 2\tilde{\mathcal{T}}^b + \Phi_h + \Phi_T + \tilde{\mathcal{K}}^b = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) - \text{div}(\tilde{H} + H)) g^\perp. \quad (49)$$

The dual equation (i.e., for $\tilde{\mathcal{D}}$ -variations) is

$$\text{div}(h - H \tilde{g}) - 2\mathcal{T}^b + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} + \mathcal{K}^b = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) - \text{div}(H + \tilde{H})) \tilde{g}. \quad (50)$$

Replacing $S_{\text{mix}} - \text{div}(H + \tilde{H})$ due to (20), we obtain (42)–(43). Replacing $\text{div} \tilde{h}$ and $\text{div} h$ in (49)–(50) due to (14)₁ and its dual (15)₁, we obtain (44)–(45). \square

Remark 1.3. Tracing (42) (and similarly for (43)), we obtain the equality

$$\mathfrak{R} - \mathfrak{R}(M, g) = 2(1 - p) \text{div} \tilde{H} + p \text{div} H,$$

where $\mathfrak{R} = p S_{\text{mix}} + 4 \|T\|^2 - 2 \|\tilde{T}\|^2 + 2 S_{\text{ex}}$. Integrating this over a closed M gives identity.

Remark 1.4. If $g \in \text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ is a critical point of the functional (3) with respect to biconformal variations, then

$$\begin{aligned} (p - 2) S_{\text{ex}} + p \tilde{S}_{\text{ex}} + (p + 2) \|T\|^2 + (p - 4) \|\tilde{T}\|^2 + 2(p - 1) \text{div} \tilde{H} \\ = p S_{\text{mix}}^*(M, g) \quad (\text{for } \mathcal{D}\text{-conformal variations}), \end{aligned} \quad (51)$$

$$\begin{aligned} (n - 2) \tilde{S}_{\text{ex}} + n S_{\text{ex}} + (n + 2) \|\tilde{T}\|^2 + (n - 4) \|T\|^2 + 2(n - 1) \text{div} H \\ = n S_{\text{mix}}^*(M, g) \quad (\text{for } \tilde{\mathcal{D}}\text{-conformal variations}). \end{aligned} \quad (52)$$

Since (52) is dual to (51), we shall show (51) only. For $S = s g^\perp$ with $s \in C^\infty(M)$ we have $\text{Tr}_g S = ps$. Applying (33) of Lemma 1.4 to (20), similarly to (46), we obtain

$$\int_M \partial_t Q(g_t)|_{t=0} d \text{vol}_g = \int_M s \left((p-1) \text{div } \tilde{H} - S_{\text{ex}} - 2 \|\tilde{T}\|_g^2 + \|T\|_g^2 \right) d \text{vol}_g. \quad (53)$$

By Proposition 1.5 and (20), we obtain

$$\begin{aligned} \frac{d}{dt} \{J_{\text{mix}}(\tilde{g}_t)\}_{t=0} &= \int_M s \left((p-1) \text{div } \tilde{H} - S_{\text{ex}} - 2 \|\tilde{T}\|_g^2 + \|T\|_g^2 \right. \\ &\quad \left. + \frac{p}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} + \|T\|_g^2 + \|\tilde{T}\|_g^2 - S_{\text{mix}}^*(M, g)) \right) d \text{vol}_g. \end{aligned} \quad (54)$$

If g is a critical point of the functional J_{mix} with respect to biconformal variations, then the integral in (54) is zero for arbitrary $s \in C^\infty(M)$, and (51) holds.

2 Applications to foliated manifolds

In this section we assume that (M, g) is a semi-Riemannian manifold endowed with a n -dimensional foliation \mathcal{F} (i.e., $T = 0$) such that $T\mathcal{F}_x$ is nondegenerate in $(T_x M, g_x)$ for every $x \in M$. Then the normal distribution \mathcal{D} is also nondegenerate in $(T_x M, g_x)$. We have $\Phi_T = 0$ and $\Psi(X, Y) = \text{Tr}_g(A_Y A_X)$. Since $\tilde{\mathcal{D}} = T\mathcal{F}$, we write $r_{\mathcal{F}} = r_{\tilde{\mathcal{D}}}$, see (8), and equations (15) read

$$r_{\mathcal{F}} = \text{div } h + \langle H, h \rangle - \mathcal{A}^b - \tilde{\Psi} + \text{Def}_{\mathcal{F}} \tilde{H}, \quad d_{\mathcal{F}} \tilde{H} = 0. \quad (55)$$

Example 2.1. Let \mathcal{F} be a one-dimensional foliation (i.e., $n = 1$) on (M^{p+1}, g) spanned by a unit vector field N , then $S_{\text{mix}} = \text{Ric}_g(N, N)$, and (3) reduces to (4). We have

$$r_{\mathcal{F}} = \text{Ric}_g(N, N) \tilde{g}, \quad r_{\mathcal{D}} = (R_N)^b,$$

where $R_N = R^\nabla(N, \cdot)N$ is the *Jacobi operator*. Let \tilde{h} be the scalar second fundamental form of \mathcal{D} . Define the functions $\tilde{\tau}_i = \text{Tr}(\tilde{A}_N^i)$ ($i \geq 0$). It is easy to see that $\tilde{S}_{\text{ex}} = \tilde{\tau}_1^2 - \tilde{\tau}_2$ and

$$\begin{aligned} \text{div } N &= \sum_i g(\nabla_i N, \mathcal{E}_i) = -g(N, \sum_i \nabla_i \mathcal{E}_i) = -\tilde{\tau}_1, \\ \text{div}(\tilde{\tau}_1 N) &= N(\tilde{\tau}_1) + \tilde{\tau}_1 \text{div } N = N(\tilde{\tau}_1) - \tilde{\tau}_1^2. \end{aligned}$$

Notice that $(H^b \otimes H^b)(X, Y) = g(H, X)g(H, Y)$. One may calculate

$$\begin{aligned} \tilde{\mathcal{A}} &= \tilde{A}_N^2, \quad g(\tilde{h}N, \tilde{H}) = \tilde{\tau}_1 \tilde{h}, \quad \Psi = H^b \otimes H^b, \quad \tilde{\Psi} = (\tilde{\tau}_2 - \|\tilde{T}\|^2) \tilde{g}, \\ \mathcal{A}^b &= \|H\|^2 \tilde{g}, \quad \mathcal{T} = 0, \quad g(h, H) = \|H\|^2 \tilde{g}, \\ H &= \nabla_N N, \quad h = H \tilde{g}, \quad \|h\| = \|H\|, \\ \tilde{H} &= \tilde{\tau}_1 N, \quad \tilde{\tau}_1 = \text{Tr}_g \tilde{h}, \quad \|\tilde{h}\|^2 = \tilde{\tau}_2, \quad \text{Def}_{\mathcal{F}} \tilde{H} = N(\tilde{\tau}_1) \tilde{g}. \end{aligned}$$

We find $\text{div}(\tilde{h}N) = \nabla_N \tilde{h} - \tilde{\tau}_1 \tilde{h}$ and $\text{div } h = (\text{div } H) \tilde{g}$. Then (14)₁ and (20) read

$$\begin{aligned} (R_N + \tilde{A}_N^2 + (\tilde{T}_N^\sharp)^2)^b &= \nabla_N \tilde{h} - H^b \otimes H^b + \text{Def}_{\mathcal{D}} H, \\ \text{Ric}_g(N, N) &= \text{div}(\tilde{\tau}_1 N + H) + \tilde{\tau}_1^2 - \tilde{\tau}_2 + \|\tilde{T}\|^2. \end{aligned} \quad (56)$$

If M is a closed manifold, then by (56) and the Divergence Theorem, we represent (4) as

$$J_{\text{mix}}(g) = \int_M (\tilde{\tau}_1^2 - \tilde{\tau}_2 + \|\tilde{T}\|^2) d \text{vol}_g. \quad (57)$$

Note that, by (12), $\text{div}(F N) = \nabla_N F - \tilde{\tau}_1 F$ for $(0, 2)$ -tensors F defined on M .

2.1 Euler-Lagrange equations for J_{mix} and critical adapted metrics

For a foliation \mathcal{F} with $T\mathcal{F} = \tilde{\mathcal{D}}$, definitions (36) have the view

$$S_{\text{mix}}^*(M, g) = \begin{cases} S_{\text{mix}}(M, g) - \frac{2}{p} (S_{\text{ex}} + 2 \|\tilde{T}\|^2)(M, g) & \text{for } \mathcal{D}\text{-variations,} \\ S_{\text{mix}}(M, g) - \frac{2}{n} (\tilde{S}_{\text{ex}} - \|\tilde{T}\|^2)(M, g) & \text{for } T\mathcal{F}\text{-variations.} \end{cases} \quad (58)$$

Using Proposition 1.7, we obtain the following.

Theorem 2.2. *Let \mathcal{F} be a foliation with a transversal distribution \mathcal{D} on a closed manifold M . If $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of J_{mix} with respect to variations (21)–(22), then the following Euler-Lagrange equations are satisfied:*

$$\text{div}(\tilde{h} - \tilde{H} g^\perp) - 2\tilde{\mathcal{T}}^\flat + \Phi_h + \tilde{\mathcal{K}}^\flat = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \quad (59)$$

$$\text{div}(h - H \tilde{g}) + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) \tilde{g} \quad (\text{for } T\mathcal{F}\text{-variations}). \quad (60)$$

Using the partial Ricci tensor, we rewrite the Euler-Lagrange equations as

$$\begin{aligned} r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \tilde{\mathcal{A}}^\flat - \tilde{\mathcal{T}}^\flat + \Phi_h + \Psi - \text{Def}_{\mathcal{D}} H + \tilde{\mathcal{K}}^\flat \\ = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \text{div}(\tilde{H} - H)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \end{aligned} \quad (61)$$

$$\begin{aligned} r_{\mathcal{F}} - \langle h, H \rangle + \mathcal{A}^\flat + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} + \tilde{\Psi} - \text{Def}_{\mathcal{F}} \tilde{H} \\ = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \text{div}(H - \tilde{H})) \tilde{g} \quad (\text{for } T\mathcal{F}\text{-variations}). \end{aligned} \quad (62)$$

Certainly, these mixed field equations admit amount of solutions (e.g., twisted products, see Sect. 2.2) which might serve as models in theoretical physics, see discussion in [2].

Corollary 2.3. *Let \mathcal{F} be a foliation spanned by a nonzero vector field N with a transversal distribution \mathcal{D} on a closed manifold M . If $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of the functional (4) with respect to variations (21), then the following Euler-Lagrange equations hold:*

$$\begin{aligned} \text{div}((\tilde{h} - \tilde{\tau}_1 g^\perp)N) - 2(\tilde{T}_N^{\sharp 2})^\flat + [\tilde{T}_N^\sharp, \tilde{A}_N]^\flat \\ = \frac{1}{2} (\tilde{\tau}_1^2 - \tilde{\tau}_2 + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \end{aligned} \quad (63)$$

$$\tilde{\tau}_1^2 - \tilde{\tau}_2 = 3 \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g) \quad (\text{for } T\mathcal{F}\text{-variations}). \quad (64)$$

One may rewrite the equations using R_N and $\text{Ric}_g(N, N)$, as

$$\begin{aligned} (R_N + \tilde{A}_N^2 - \tilde{T}_N^{\sharp 2})^\flat - \tilde{\tau}_1 \tilde{h} + H^\flat \otimes H^\flat - \text{Def}_{\mathcal{D}} H \\ = \frac{1}{2} (\text{Ric}_g(N, N) - S_{\text{mix}}^*(M, g) + \text{div}(\tilde{\tau}_1 N - H)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \end{aligned} \quad (65)$$

$$\text{Ric}_g(N, N) = -S_{\text{mix}}^*(M, g) + 4 \|\tilde{T}\|^2 + \text{div}(\tilde{\tau}_1 N + H) \quad (\text{for } T\mathcal{F}\text{-variations}). \quad (66)$$

Proof. Substituting the values $\Phi_h = 0 = S_{\text{ex}}$, $\tilde{S}_{\text{ex}} = \tilde{\tau}_1^2 - \tilde{\tau}_2$ and $\tilde{\mathcal{T}} = \tilde{T}_N^{\sharp 2}$ into (59) yields (63). Substituting the values $h = H \tilde{g}$, $\Phi_{\tilde{h}} = (\tilde{\tau}_1^2 - \tilde{\tau}_2) \tilde{g}$ and $\Phi_{\tilde{T}} = -\|\tilde{T}\|^2 \tilde{g}$ into (60) yields (64). Then, replacing terms in (63) and (64) due to (56), we obtain (65) and (66). \square

Example 2.4. (i) Let \mathcal{F} be a totally umbilical foliation (i.e., $h = \frac{1}{n} H \tilde{g}$, $T = 0$). Thus,

$$\Phi_h = \frac{n-1}{n} H^\flat \otimes H^\flat, \quad \mathcal{A}^\flat = \frac{1}{n^2} \|H\|^2 \tilde{g}, \quad \Psi = \frac{1}{n} H^\flat \otimes H^\flat, \quad S_{\text{ex}} = \frac{n-1}{n} \|H\|^2$$

hold, and the fundamental equations (14)₁ and (55)₁ read

$$r_{\mathcal{D}} - \operatorname{div} \tilde{h} - \langle \tilde{H}, \tilde{h} \rangle + (\tilde{\mathcal{A}} - \tilde{\mathcal{T}})^b + \frac{1}{n} H^b \otimes H^b - \operatorname{Def}_{\mathcal{D}} H = 0, \quad (67)$$

$$r_{\mathcal{F}} - \frac{1}{n} \left(\operatorname{div} H + \frac{n-1}{n} \|H\|^2 \right) \tilde{g} + \tilde{\Psi} - \operatorname{Def}_{\mathcal{F}} \tilde{H} = 0. \quad (68)$$

(ii) Let \mathcal{F} be a totally geodesic foliation (i.e., $h = 0 = T$) of a closed Riemannian manifold (M, g) . Thus, $S_{\text{ex}} = 0$, and (14)₁ and (55)₁ read

$$r_{\mathcal{D}} - \operatorname{div} \tilde{h} - \langle \tilde{H}, \tilde{h} \rangle + (\tilde{\mathcal{A}} + \tilde{\mathcal{T}})^b = 0, \quad r_{\mathcal{F}} - \operatorname{Def}_{\mathcal{F}} \tilde{H} + \tilde{\Psi} = 0.$$

If $g \in \operatorname{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of J_{mix} with respect to \mathcal{D} -variations, then, see Euler-Lagrange equations (59) and (61),

$$\operatorname{div}(\tilde{h} - \tilde{H} g^\perp) - 2\tilde{\mathcal{T}}^b + \tilde{\mathcal{K}}^b = \frac{1}{2} (\tilde{S}_{\text{ex}} + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) g^\perp, \quad (69)$$

$$r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \tilde{\mathcal{A}}^b - \tilde{\mathcal{T}}^b + \tilde{\mathcal{K}}^b = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \operatorname{div} \tilde{H}) g^\perp. \quad (70)$$

(iii) Let \mathcal{F} be a Riemannian foliation (i.e., $\tilde{h} = 0 = T$) of a closed manifold (M, g) . Note that $\tilde{S}_{\text{ex}} = 0$. If $g \in \operatorname{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of J_{mix} with respect to $T\mathcal{F}$ -variations, then, see Euler-Lagrange equations (60) and (62),

$$\operatorname{div}(h - H \tilde{g}) + \Phi_{\tilde{T}} = \frac{1}{2} (S_{\text{ex}} + \|\tilde{T}\|^2 - S_{\text{mix}}^*(M, g)) \tilde{g}, \quad (71)$$

$$r_{\mathcal{F}} + \Phi_{\tilde{T}} + \tilde{\Psi} = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(M, g) + \operatorname{div} H) \tilde{g}. \quad (72)$$

Next we consider sufficient conditions for critical metrics of the action (3).

Corollary 2.5. *Let \mathcal{F} be a foliation on a closed Riemannian manifold (M, g) with integrable normal distribution \mathcal{D} . Then*

$$\operatorname{div}(\tilde{h} - \tilde{H} g^\perp) + \Phi_h = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(M, g)) g^\perp \quad (\text{for } \mathcal{D}\text{-variations}), \quad (73)$$

$$\operatorname{div}(h - H \tilde{g}) + \Phi_{\tilde{h}} = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(M, g)) \tilde{g} \quad (\text{for } T\mathcal{F}\text{-variations}), \quad (74)$$

where, see (58),

$$S_{\text{mix}}^*(M, g) = \begin{cases} S_{\text{mix}}(M, g) - \frac{2}{p} S_{\text{ex}}(M, g) & \text{for } \mathcal{D}\text{-variations,} \\ S_{\text{mix}}(M, g) - \frac{2}{n} \tilde{S}_{\text{ex}}(M, g) & \text{for } T\mathcal{F}\text{-variations.} \end{cases} \quad (75)$$

(i) If $g \in \operatorname{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of J_{mix} with respect to \mathcal{D} -variations and \mathcal{F} is totally geodesic then $\operatorname{div}(\tilde{h} - \frac{1}{p} \tilde{H} g^\perp) = 0$.

(ii) If $g \in \operatorname{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of J_{mix} with respect to $T\mathcal{F}$ -variations and \mathcal{F} is totally umbilical then $\Phi_{\tilde{h}} = \frac{1}{n} \tilde{S}_{\text{ex}} \tilde{g}$.

Proof. From (59)–(60) with $\tilde{T} = 0$ we obtain (73)–(74). (i) Tracing (73) with $\Phi_h = 0 = S_{\text{ex}}$, we find $\tilde{S}_{\text{ex}} - S_{\text{mix}}^*(M, g) = 2 \frac{1-p}{p} \operatorname{div} \tilde{H}$. From this and (73) the claim follows. (ii) Similarly, tracing (74) with $h = \frac{1}{n} H \tilde{g}$, we find $S_{\text{ex}} + \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(M, g) = 2 \frac{1-n}{n} \operatorname{div} H + \frac{2}{n} \tilde{S}_{\text{ex}}$. From this and (74) the claim follows. \square

Remark 2.1. By Corollary 2.5, if \mathcal{F} is a totally geodesic foliation on a closed Riemannian manifold (M, g) with integrable normal distribution \mathcal{D} , and $g \in \operatorname{Riem}(M, T\mathcal{F}, \mathcal{D})$ is critical for the action J_{mix} with respect to all adapted variations of (M, g) then

$$\operatorname{div} \left(\tilde{h} - \frac{1}{p} \tilde{H} g^\perp \right) = 0, \quad \Phi_{\tilde{h}} = \frac{1}{n} \tilde{S}_{\text{ex}} \tilde{g}. \quad (76)$$

Obviously, (76)₁ is satisfied when \mathcal{D} is totally umbilical, i.e., $\tilde{h} = \frac{1}{p} \tilde{H} g^\perp$. On the other hand, (76)₂ is not satisfied when \mathcal{D} is totally umbilical (but not totally geodesic).

Problem: *Classify closed Riemannian manifolds admitting a totally geodesic foliation with integrable normal distribution satisfying (any of) conditions (76).*

2.2 Double-twisted products of manifolds

Let $M = M_1 \times M_2$ be a product of semi-Riemannian manifolds (M_i, g_i) ($i \in \{1, 2\}$). Let $\pi_i : M \rightarrow M_i$ and $P_i = d\pi_i : TM \rightarrow TM_i$ be the canonical projections. Given twisting functions $f_i \in C^\infty(M)$ a *double-twisted product* $M_1 \times_{(f_1, f_2)} M_2$ is $M_1 \times M_2$ with the metric

$$g = e^{f_1} \pi_1^* g_1 + e^{f_2} \pi_2^* g_2. \quad (77)$$

If $f_1 = \text{const}$ then we have a *twisted product* (a *warped product* if, in addition, $f_2 = F \circ \pi_1$ for some $F \in C^\infty(M_1)$). Both families, the *leaves* $M_1 \times \{y\}$ and the *fibers* $\{x\} \times M_2$, are totally umbilical in (M, g) and this property characterizes double-twisted products (cf. R. Ponge & H. Reckziegel, [13]). We have $T = \tilde{T} = 0$ and

$$\begin{aligned} A_Y &= -Y(f_1) \text{id}, & h &= -(\nabla^\perp f_1) \tilde{g}, & H &= -n \nabla^\perp f_1, \\ \tilde{A}_X &= -X(f_2) \text{id}^\perp, & \tilde{h} &= -(\tilde{\nabla} f_2) g^\perp, & \tilde{H} &= -p \tilde{\nabla} f_2, \end{aligned}$$

where $X \in \tilde{\mathcal{D}}$ and $Y \in \mathcal{D}$ are unit vectors. In this case, cf. (11),

$$\text{div } \tilde{H} = -p \tilde{\Delta} f_2 - p^2 \|\tilde{\nabla} f_2\|^2, \quad \text{div } H = -n \Delta^\perp f_1 - n^2 \|\nabla^\perp f_1\|^2.$$

For any double-twisted product, we have identities, see (68),

$$r_{\mathcal{D}} = \frac{1}{p} \left(\text{div } \tilde{H} + \frac{p-1}{p} \|\tilde{H}\|^2 \right) g^\perp + \text{Def}_{\mathcal{D}} H - \frac{1}{n} H^\flat \otimes H^\flat, \quad (78)$$

$$r_{\mathcal{F}} = \frac{1}{n} \left(\text{div } H + \frac{n-1}{n} \|H\|^2 \right) \tilde{g} + \text{Def}_{\mathcal{F}} \tilde{H} - \frac{1}{p} \tilde{H}^\flat \otimes \tilde{H}^\flat. \quad (79)$$

Hence, $S_{\text{mix}} = \text{div}(H + \tilde{H}) + \frac{n-1}{n} \|\tilde{H}\|^2 + \frac{p-1}{p} \|H\|^2$ and $J_{\text{mix}}(g) \geq 0$.

The leaves $M_1 \times \{y\}$ of a twisted product are totally geodesic submanifolds on M (i.e., $h = 0$). For a twisted product (i.e., $f_2 = 0$), we have identities, see (78) and (79),

$$r_{\mathcal{D}} = \frac{1}{p} \left(\text{div } \tilde{H} + \frac{p-1}{p} \|\tilde{H}\|^2 \right) g^\perp, \quad r_{\mathcal{F}} = \text{Def}_{\mathcal{F}} \tilde{H} - \frac{1}{p} \tilde{H}^\flat \otimes \tilde{H}^\flat.$$

Corollary 2.6. *A double-twisted product metric (77) on a closed manifold $M = M_1^n \times M_2^p$ with $n, p > 1$ is a critical point of the functional (3) with respect to \mathcal{D} -variations if and only if $f_1 = \text{const}$ (i.e., a twisted product). Similarly, in the case of $T\mathcal{F}$ -variations, we get $f_2 = \text{const}$.*

Proof. Let the metric g be critical with respect to \mathcal{D} -variations. Since $\tilde{h} = \frac{1}{p} \tilde{H} g^\perp$ and $\Phi_h = \frac{n-1}{n} H^\flat \otimes H^\flat$, by (59), we have

$$\frac{1-p}{p} (\text{div } \tilde{H}) g^\perp + \frac{n-1}{n} H^\flat \otimes H^\flat = \frac{1}{2} \left(\frac{n-1}{n} \|H\|^2 + \frac{p-1}{p} \|\tilde{H}\|^2 - S_{\text{mix}}^*(M, g) \right) g^\perp. \quad (80)$$

Since the symmetric tensor $H^\flat \otimes H^\flat$ has rank ≤ 1 , then for $p > 1$, we obtain $H = 0$; in this case, (80) becomes identity. This corresponds to totally geodesic leaves, i.e., the twisted product. The converse claim is also true. The case of $T\mathcal{F}$ -variations is similar. \square

By Corollary 2.6, among double-twisted products, the (biscaling of) direct product metrics are only critical points of the functional J_{mix} with respect to all adapted variations.

Example 2.7 (Foliations of the standard sphere omitting a codimension 2 totally geodesic submanifold). Let $S^m = \{x \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} x_i^2 = 1\}$ be the unit sphere in \mathbb{R}^{m+1} and $\Sigma \subset S^m$ a codimension 2 totally geodesic submanifold. Then (by an appropriate choice of coordinates on \mathbb{R}^{m+1}), $\Sigma = \{x \in S^m : x_1 = x_2 = 0\}$. Let S_+^{m-1} be the hemisphere $\{y = (y', y_m) \in S^{m-1} : y_m > 0\}$, where $y' = (y_1, \dots, y_{m-1})$, and \mathcal{F}_Σ be the foliation of $S^m \setminus \Sigma$, whose leaf space is

$$(S^m \setminus \Sigma) / \mathcal{F}_\Sigma = \{L_\zeta : \zeta \in S^1\}, \quad L_\zeta \equiv \{(y', \operatorname{Re}(\zeta) y_m, \operatorname{Im}(\zeta) y_m) : y \in S_+^{m-1}\}.$$

Let $\tilde{\mathcal{D}} = T\mathcal{F}_\Sigma$ and \mathcal{D} be its g_m -orthogonal complement with respect to the canonical Riemannian metric $g_m \in \operatorname{Riem}(S^m \setminus \Sigma)$. Then g_m is a \mathcal{D} -critical point of the functional (3).

To prove the claim, note that $S^m \setminus \Sigma$ is isometric to the warped product $S_+^{m-1} \times_w S^1$ with the warping function $w(y) = y_m$. Precisely let g_N be the first fundamental form of S^N in the Euclidean space \mathbb{R}^{N+1} . Then the map $I : S_+^{m-1} \times S^1 \rightarrow S^m \setminus \Sigma$ given by

$$I(y, \zeta) = (y', \operatorname{Re}(\zeta) y_m, \operatorname{Im}(\zeta) y_m), \quad y \in S_+^{m-1}, \quad \zeta \in S^1,$$

is an isometry of $S_+^{m-1} \times S^1$ with the warped product metric $\pi_1^* g_{m-1} + (w \circ \pi_1)^2 \pi_2^* g_1$ onto $(S^m \setminus \Sigma, g_m)$. Here $\pi_1 : S_+^{m-1} \times S^1 \rightarrow S_+^{m-1}$ and $\pi_2 : S_+^{m-1} \times S^1 \rightarrow S^1$ are the projections. Let \mathcal{F}_+ be the foliation of $S_+^{m-1} \times S^1$ whose leaves are $(S_+^{m-1} \times S^1) / \mathcal{F}_+ = \{S_+^{m-1} \times \{\zeta\} : \zeta \in S^1\}$ so that I is a foliated map of $(S_+^{m-1} \times S^1, \mathcal{F}_+)$ onto $(S^m \setminus \Sigma, \mathcal{F}_\Sigma)$. Finally (by Corollary 2.6) $\pi_1^* g_{m-1} + (w \circ \pi_1)^2 \pi_2^* g_1$ is $(T\mathcal{F}_+)^{\perp}$ -critical.

2.3 Codimension-one foliations

Let \mathcal{F} be a codimension one foliation (i.e., $p = 1$) of a closed manifold (M^{n+1}, g) , and let the normal subbundle \mathcal{D} be spanned by a unit field $N \in \mathfrak{X}_M$. We have $T = 0 = \tilde{T}$ and

$$h(X, Y) = g(\nabla_X Y, N), \quad A_N(X) = -\nabla_X N, \quad \mathcal{A} = A_N^2 \quad (X, Y \in T\mathcal{F}),$$

where h is the scalar second fundamental form and A_N the Weingarten operator of \mathcal{F} . Note that $\tilde{H} = \nabla_N N$ is the curvature vector of N -curves, $\tilde{h} = \tilde{H}g^{\perp}$, $\tilde{\Psi} = \tilde{H}^{\flat} \otimes \tilde{H}^{\flat}$ and

$$\tilde{A}_Y(N) = g(\tilde{H}, Y)N, \quad \tilde{\mathcal{A}} = \sum_a g(\tilde{H}, E_a)^2 N = \|\tilde{H}\|^2 N \quad (Y \in T\mathcal{F}).$$

We also have, see (7) and (8): $r_{\mathcal{D}} = \operatorname{Ric}_g(N, N)g^{\perp}$ and $r_{\mathcal{F}} = (R_N)^{\flat}$.

Power sums of the principal curvatures k_1, \dots, k_n (the eigenvalues of A_N) are given by [18]

$$\tau_{\alpha} = k_1^{\alpha} + \dots + k_n^{\alpha} = \operatorname{Tr}(A_N^{\alpha}), \quad \alpha \geq 0.$$

The τ 's can be expressed using the elementary symmetric functions of the eigenvalues of A_N (called *mean curvatures* in the literature),

$$\sigma_0 = 1, \quad \sigma_a = \sum_{i_1 < \dots < i_a} k_{i_1} \cdot \dots \cdot k_{i_a} \quad (1 \leq a \leq n).$$

For example, $\sigma_1 = \tau_1 = \operatorname{Tr} A_N$, and $2\sigma_2 = \tau_1^2 - \tau_2$ when $n > 1$ (and $\tau_1^2 - \tau_2 = 0$ when $n = 1$). The equality $\operatorname{div} N = -\tau_1$ holds and $H = \tau_1 N$ is the mean curvature vector of \mathcal{F} .

Note that $S_{\text{ex}} = \tau_1^2 - \tau_2$ and definitions (58) read

$$S_{\text{mix}}^*(M, g) = \begin{cases} S_{\text{mix}}(M, g) - 2S_{\text{ex}}(M, g) & \text{for } \mathcal{D}\text{-variations,} \\ S_{\text{mix}}(M, g) & \text{for } T\mathcal{F}\text{-variations.} \end{cases} \quad (81)$$

By (55)₁ (see also (56)), we obtain

$$(R_N + A_N^2)^{\flat} = \nabla_N h - \tilde{H}^{\flat} \otimes \tilde{H}^{\flat} + \operatorname{Def}_{\mathcal{F}} \tilde{H}. \quad (82)$$

Then we find (tracing (82) or by (56)₂ with $T = 0$) (cf. also [18, 24])

$$\text{Ric}_g(N, N) = \text{div}(\tau_1 N + \tilde{H}) + \tau_1^2 - \tau_2. \quad (83)$$

Since M is a closed manifold, by (83) and the Divergence Theorem, we represent (4) as

$$J_{\text{mix}}(g) = \int_M (\tau_1^2 - \tau_2) \, d \, \text{vol}_g; \quad (84)$$

thus, $S_{\text{mix}}(M, g) = S_{\text{ex}}(M, g)$. By (12), $\text{div}(F N) = \nabla_N F - \tau_1 F$ for $(0, 2)$ -tensors F on M .

Next, we calculate the gradient of J_{mix} with respect to adapted variations of a metric (see [18, § 2.3.3] for $T\mathcal{F}$ -variations).

Corollary 2.8 (of Theorem 2.2). *Let \mathcal{F} be a codimension one foliation on a closed manifold M^{n+1} , whose transversal distribution \mathcal{D} is spanned by a nonzero vector field N . If $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ is a critical point of the functional (4) with respect to variations (21), then the following Euler-Lagrange equations are satisfied:*

$$\tau_1^2 - \tau_2 = -S_{\text{mix}}^*(M, g) \quad (\text{for } \mathcal{D}\text{-variations}), \quad (85)$$

$$\text{div}((h - \tau_1 \tilde{g})N) = \frac{1}{2}(\tau_1^2 - \tau_2 - S_{\text{mix}}^*(M, g)) \tilde{g} \quad (\text{for } T\mathcal{F}\text{-variations}). \quad (86)$$

Proof. From (64) and (63) with $p \leftrightarrow n$ and then using $T = 0$ we have (85) and (86). \square

Remark 2.2. (i) By (85), if $n = 1$ then $J_{\text{mix}}(g) = 0$, and if $n > 1$, then $\sigma_2 = \text{const}$, resp., $\sigma_2 = 0$ when variations do not preserve the volume of (M, g) . Replacing terms in (85)–(86) due to (82)–(83) (or using (65) and (66) with $\tilde{T} = 0$), we rewrite the Euler-Lagrange equations as

$$\text{Ric}_g(N, N) = \frac{1}{2}(\text{Ric}_g(N, N) - S_{\text{mix}}^*(M, g) + \text{div}(\tau_1 N + \tilde{H})) \quad (\text{for } \mathcal{D}\text{-variations}), \quad (87)$$

$$\begin{aligned} (R_N + A_N^2)^b - \tau_1 h + \tilde{H}^b \otimes \tilde{H}^b - \text{Def}_{\mathcal{F}} \tilde{H} \\ = \frac{1}{2}(\text{Ric}_g(N, N) - S_{\text{mix}}^*(M, g) + \text{div}(\tau_1 N - \tilde{H})) \tilde{g} \quad (\text{for } T\mathcal{F}\text{-variations}). \end{aligned} \quad (88)$$

(ii) Equation (86) for $T\mathcal{F}$ -variations (21)–(22) is equivalent to [18, Example 2.5], where notation $J_{\text{mix}} = 2 E_N$ is used and the Euler-Lagrange equations are given in the form

$$-\text{div}(T_1(h)N) = \frac{1}{2}(\tau_1^2 - \tau_2 - S_{\text{mix}}^*(M, g)) \tilde{g}. \quad (89)$$

Here, $T_1(h) = \tau_1 \tilde{g} - h$ is the first Newton transformation of h . By the above,

$$-\text{div}(T_1(h)N) = \nabla_N h - \tau_1 h - \text{div}(\tau_1 N) \tilde{g};$$

hence, (89) reduces to (86).

Since $\Phi_{\tilde{h}} = 0 = \tilde{S}_{\text{ex}}$, from Proposition 2.5(ii) we have the following.

Proposition 2.9. *Let \mathcal{F} be a codimension-one foliation of a closed Riemannian manifold (M^{n+1}, g) ($n > 1$) with normal distribution spanned by a unit vector field N . Then g is a critical point of J_{mix} with respect to adapted variations preserving the volume of (M, g) if and only if*

$$\tau_1 = 0, \quad \sigma_2 = \text{const} \leq 0, \quad \nabla_N h = 0. \quad (90)$$

In particular, (90) are satisfied trivially when \mathcal{F} is totally geodesic.

Proof. By (85) and (81), we have $S_{\text{ex}} = S_{\text{mix}}(M, g)$, hence $\sigma_2 = -\tau_2 = \text{const} \leq 0$. Tracing (86) and using (81), we obtain

$$(1 - n)(N(\tau_1) - \tau_1^2) = \frac{n}{2}(\tau_1^2 - \tau_2 - S_{\text{mix}}(M, g))$$

Since $n > 1$, we obtain the ODE $N(\tau_1) = \tau_1^2$ on complete N -curves, which has trivial solution $\tau_1 = 0$. Then, by (86), we have $\nabla_N h = 0$. \square

Corollary 2.10. *Let \mathcal{F} be a 2-dimensional foliation on a closed 3-dimensional Riemannian manifold (M, g) with normal distribution spanned by a unit vector field N . Then g is a critical point of J_{mix} with respect to adapted variations preserving the volume of (M, g) if and only if*

- 1) *the principal curvatures k_1, k_2 of the leaves are constant on M and $k_1 + k_2 = 0$;*
- 2) *the corresponding eigenvectors E_1 and E_2 are parallel in the N -direction.*

In particular, M is parallelizable: $\{E_1, E_2, N\}$ is the global orthonormal frame.

Example 2.11 (Foliations by level hypersurfaces). Let (M^{n+1}, g) be a Riemannian manifold, $u \in C^\infty(M, \mathbb{R})$, and let \mathcal{F} be the foliation of $U = M \setminus \text{Crit}(u)$ by level hypersurfaces of u . Then $N = \lambda^{-1} \nabla u$, where $\lambda \equiv \|\nabla u\|$ and $\tau_1 = -\text{div } N = -\lambda^{-1} \text{div}(\nabla u) - (\nabla u)(\lambda^{-1})$, i.e.,

$$\tau_1 = -\lambda^{-1} \Delta u + \lambda^{-2} (\nabla u)(\lambda). \quad (91)$$

The geometric background needed to write the Euler-Lagrange equations of Corollary 2.8 is presented in [23, p. 104–116]. We need to rephrase a few facts in [23] under the conventions adopted in this paper. Let $\pi : TU \rightarrow \nu(\mathcal{F}) \equiv TU/T\mathcal{F}$ be the transverse bundle. Let h be the second fundamental form of \mathcal{F} in (U, g) , i.e., $h(X, Y) = \pi(\nabla_X Y)$ for any $X, Y \in \mathfrak{X}_{\mathcal{F}}$. Then

$$h(X, Y) = -\lambda^{-1} \text{Hess}_u(X, Y) \pi(N) \quad (92)$$

by [23, (8.5), p. 105]. Here Hess_u is the Hessian of u , i.e., $\text{Hess}_u(V, W) = (\nabla_V du)W$ for any $V, W \in \mathfrak{X}_U$. Let $\sigma : \nu(\mathcal{F}) \rightarrow T^\perp \mathcal{F} \equiv \mathcal{D}$ be the natural bundle isomorphism, i.e., $\sigma(s) = V^\perp$ for any $s \in \nu(\mathcal{F})$ and any $V \in \mathfrak{X}_U$ such that $\pi(V) = s$. Here V^\perp is the \mathcal{D} -component of V with respect to the decomposition $TU = T\mathcal{F} \oplus \mathcal{D}$. The metric g_Q induced by g on the transverse bundle is $g_Q(r, s) = g(\sigma(r), \sigma(s))$ for any $r, s \in Q \equiv \nu(\mathcal{F})$. The Weingarten operator $A_N : T\mathcal{F} \rightarrow T\mathcal{F}$ is given by

$$g(A_N(X), Y) = g_Q(h(X, Y), \pi(N)).$$

Then (by (92))

$$g(A_N X, Y) = -\lambda^{-1} \text{Hess}_u(X, Y). \quad (93)$$

Let $\nu = \lambda^{-1} du$ be the transverse volume element. Let $\{E_a : 1 \leq a \leq n\}$ be a local orthonormal frame of $T\mathcal{F}$ and $\{\omega^a : 1 \leq a \leq n\}$ the dual coframe, i.e., $\omega^i(E_j) = \delta_j^i$ and $\omega^i(N) = 0$. Then $\{E_a, N : 1 \leq a \leq n\}$ is a local orthonormal frame of TU and $\{\omega^a, \nu : 1 \leq a \leq n\}$ is the corresponding dual coframe, and (93) may be locally written

$$A_N X = -\lambda^{-1} \sum_a \text{Hess}_u(X, E_a) E_a. \quad (94)$$

Then (by (94))

$$\tau_2 = \text{Tr}(A_N^2) = \sum_a g(A_N E_a, A_N E_a) = \lambda^{-2} \sum_{a,b} (\text{Hess}_u(E_a, E_b))^2,$$

so that (by $\text{Hess}_u(N, N) = N^2(u)$)

$$\tau_2 = \lambda^{-2} \left\{ \|\text{Hess}_u\|^2 - (N^2(u))^2 \right\}. \quad (95)$$

Note that $N(u) = \lambda$; hence,

$$N^2(u) = \frac{1}{\|\nabla u\|} g(\nabla u, \nabla(\|\nabla u\|)). \quad (96)$$

Then (by (84), (91) and (95)-(96))

$$\begin{aligned} J_{\text{mix}}(g) = & \int_M \left\{ \frac{1}{\|\nabla u\|^2} \left(\Delta u - \frac{1}{\|\nabla u\|} g(\nabla f, \nabla(\|\nabla f\|)) \right)^2 \right. \\ & \left. - \frac{1}{\|\nabla u\|^2} \left(\|\text{Hess}_u\|^2 - \frac{1}{\|\nabla u\|} g(\nabla u, \nabla(\|\nabla u\|)) \right) \right\} d\text{vol}_g. \end{aligned} \quad (97)$$

By (85), a metric g on U is critical for (97) with respect to \mathcal{D} -variations if $\tau_1^2 - \tau_2 = 0$, that is trivial for $n = 1$, or (by (91) and (95)) $(\Delta u - \lambda^{-1}(\nabla u)(\lambda))^2 = \|\text{Hess}_u\|^2 - N^2(u)$ that is (by (96))

$$\left(\Delta u - \frac{(\nabla u)(\|\nabla u\|)}{\|\nabla u\|} \right)^2 = \|\text{Hess}_u\|^2 - \left(\frac{(\nabla u)(\|\nabla u\|)}{\|\nabla u\|} \right)^2. \quad (98)$$

Example 2.12 (Foliations in thermodynamics). Let $M \subset \mathbb{R}^k$ be the phase space of a thermodynamical system. Let $P, V, E \in C^\infty(M, \mathbb{R})$ be respectively the pressure, volume and internal energy. If $\omega = dE + P dV \in \Omega^1(M)$ then $\omega \wedge d\omega = 0$ by the second principle of thermodynamics (cf. e.g. [6]), i.e. the distribution $\tilde{\mathcal{D}} = \text{Ker}(\omega)$ is involutive. Let \mathcal{F} be the foliation of M such that $T(\mathcal{F}) = \tilde{\mathcal{D}}$. Let $g \in \text{Riem}(M)$. If $\lambda = \|\omega\|$ then $N = \lambda^{-1}(\nabla E + P \nabla V)$. Next

$$A_N = -\frac{1}{\lambda} (\nabla \nabla E + (dP) \otimes \nabla V + P \nabla \nabla V) + \frac{1}{\lambda^2} (d\lambda) \otimes (\nabla E + P \nabla V)$$

so that

$$\begin{aligned} \tau_1 = & -\lambda^{-1} (\Delta E + (\nabla V)(P) + P \Delta V) + \lambda^{-2} ((\nabla E)(\lambda) + P(\nabla V)(\lambda)), \\ \tau_2 = & \lambda^{-2} (\|\mathcal{E}\|^2 + 2P g^*(\mathcal{E}, \mathcal{V}) + P^2 \|\mathcal{V}\|^2 + \|\nabla V\|^2 \|\nabla P\|^2) + \\ & + 2\lambda^{-2} (g(\nabla_{\nabla P} \nabla E, \nabla V) + P g(\nabla_{\nabla P} \nabla V, \nabla V)) + \\ & + \lambda^{-4} \|\nabla \lambda\|^2 (\|\nabla E\|^2 + 2P g(\nabla E, \nabla V) + P^2 \|\nabla V\|^2), \end{aligned}$$

where we have set $\mathcal{V}X = \nabla_X \nabla V$ and $\mathcal{E}X = \nabla_X \nabla E$ for every $X \in \mathfrak{X}_{\mathcal{F}}$. All processes in a simple mechanically isolated thermodynamic system are isochoric, i.e., $V = \text{const}$. If this is the case \mathcal{F} is the foliation of $M \setminus \text{Crit}(E)$ by surfaces of constant internal energy, a situation covered by Example 2.11, i.e. a metric g is critical with respect to \mathcal{D} -variations if $\Delta E = f_{\pm}(E)$, where $f_{\pm}(E) = \|\nabla E\|^{-1}(\nabla E)(\|\nabla E\|) \pm [\|\nabla \nabla E\|^2 + \|\nabla E\|^{-2} \|\nabla(\|\nabla E\|)\|^2]^{\frac{1}{2}}$.

Example 2.13 (Foliations by level lines on open Riemann surfaces). Let M be a closed surface, then $J_{\text{mix}}(g) = 2\pi\chi(M)$ for any $g \in \text{Riem}(M)$. Hence, $DJ_{\text{mix}}(g) = 0$ (any metric is critical). However, if M is an open Riemann surface, the problem of looking for stationary metrics is well posed and nontrivial. Here, one works with a known functional, because $S_{\text{mix}}(g)$ is merely the Gaussian curvature of the metric g , as observed in the introduction to this paper, yet its domain is the variety $\text{Riem}(M, \tilde{\mathcal{D}}, \mathcal{D})$ depending on the given line fields $(\tilde{\mathcal{D}}, \mathcal{D})$ on M . Cohn-Vossen first proved that the total curvature of a finitely connected complete noncompact Riemannian surface M is bounded above by $2\pi\chi(M)$. The total curvature of such a manifold M is not a topological invariant but is dependent upon the choice of a Riemannian metric, see [22].

The results in Example 2.11 apply to foliations by level lines on real surfaces (as studied by R. Jerrard & L. Rubel, [8]) associated to harmonic functions (e.g. real parts of holomorphic functions, cf. also [7]). Let $D \subset \mathbb{C}$ be a relatively compact domain and $f = u + iv$ a holomorphic function, where $u, v : D \rightarrow \mathbb{R}$ its real and imaginary parts. Let \mathcal{F} be the foliation of $U = D \setminus \text{Crit}(u)$ by level lines of u and \mathcal{D} is spanned by ∇u .

If, for instance, $f(z) = -iz$ with $z \in \mathbb{C}$ then, for an arbitrary metric $g \in \text{Riem}(\mathbb{C})$,

$$\nabla u = g^{i2} \frac{\partial}{\partial x^i}, \quad \lambda = \sqrt{g^{22}}, \quad (\nabla u)(\lambda) = \frac{1}{2\lambda} (g^{12} (g^{22})_x + g^{22} (g^{22})_y),$$

$$\Delta u = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{i2}), \quad \text{Hess}_u \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = - \left\{ \begin{matrix} 2 \\ ij \end{matrix} \right\},$$

$$\|\text{Hess}_u\|^2 = g^{ij} g^{kl} \left\{ \begin{matrix} 2 \\ ik \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ jl \end{matrix} \right\},$$

$$\begin{aligned} S_{\text{mix}} = K &= \frac{1}{\lambda^2} \left\{ \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{i2}) - \frac{1}{2\lambda^2} (g^{12} (g^{22})_x + g^{22} (g^{22})_y) \right\}^2 - \\ &- \frac{1}{\lambda^2} \left\{ g^{ij} g^{kl} \left\{ \begin{matrix} 2 \\ ik \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ jl \end{matrix} \right\} - \frac{1}{2\lambda^2} (g^{12} (g^{22})_x + g^{22} (g^{22})_y) \right\}, \end{aligned}$$

where $G = \det[g_{ij}]$. Let $\Omega \subset \mathbb{C}$. By Corollary 2.8, an adapted metric $g \in \text{Riem}(\mathbb{C})$ with $K(g) = 0$ (that implies $K(M, g) = 0$) is \mathcal{D} -critical (where $\mathcal{D} = \mathbb{R}\nabla u$).

Euclidean metric $g_0 = dx^2 + dy^2$ is trivially a \mathcal{D} -critical point. Let us look for critical metrics conformal to Euclidean metric, i.e., $g = e^\phi g_0$ with $\phi \in C^\infty(\mathbb{C})$. Then

$$g^{ij} = e^{-\phi} \delta^{ij}, \quad G = e^{2\phi}, \quad \lambda = e^{-\phi/2}, \quad \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} (\phi_{|k} \delta_j^i + \phi_{|j} \delta_k^i - \delta_{jk} \phi_{|i}),$$

$$\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{i2}) = 0, \quad g^{ij} g^{kl} \left\{ \begin{matrix} 2 \\ ik \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ jl \end{matrix} \right\} = \frac{1}{2} e^{-2\phi} (\phi_x^2 + \phi_y^2),$$

$$g^{12} (g^{22})_x + g^{22} (g^{22})_y = -e^{-2\phi} \phi_y, \quad K(e^\phi g_0) = -\frac{1}{2} e^{-\phi} \left(\phi_x^2 + \frac{1}{2} \phi_y^2 + e^\phi \phi_y \right);$$

hence, g is \mathcal{D} -critical when ϕ is a solution to the PDE

$$\phi_x^2 + \phi_y^2/2 + e^\phi \phi_y = 0. \quad (99)$$

For instance, the nonconstant solutions ϕ to (99) with $\phi_x = 0$ are $\phi(y) = -\log|2y + c|$, $c \in \mathbb{R}$. Consequently, if \mathcal{F} is the model foliation by lines parallel to the x -axis, the adapted metrics $g_c = (2y + c)^{-1} (dx^2 + dy^2)$, $c \in \mathbb{R}$, are \mathcal{D} -critical on the half-plane $U_c \equiv \{(x, y) \in \mathbb{R}^2 : y > -c/2\}$. Similarly if $f(z) = z$ (producing the foliation of \mathbb{R}^2 by lines parallel to the y -axis) the metrics $g = (2x + c)^{-1} \{dx^2 + dy^2\}$ are \mathcal{D} -critical.

Let us go back to foliations \mathcal{F} of $U = D \setminus \text{Crit}(u)$ by level lines of u . The unit normal to \mathcal{F} in Euclidean plane (\mathbb{R}^2, g_0) is $N = \lambda^{-1} (u_x \partial_x + u_y \partial_y)$ with $\lambda = (u_x^2 + u_y^2)^{\frac{1}{2}}$. Let $\kappa \in T_0^1 U$ be given by $X \lrcorner \kappa = 0$ and $\kappa(Z) = \text{Tr}(A_Z)$ for any $X \in \mathfrak{X}_{\tilde{\mathcal{D}}}$ and $Z \in \mathfrak{X}_{\mathcal{D}}$. Then $\tau_1 = \kappa(N)$. By a result in [8] (cf. also [23, p. 112])

$$\tau_1 = \lambda^{-3} \mathcal{L}(u), \quad (100)$$

where $\mathcal{L} : u \rightarrow (u_x^2 - u_y^2) u_{xx} + 2u_x u_y u_{xy}$ is the nonlinear operator. Moreover, $\|\text{Hess}_u\|^2 = 2(u_{xx}^2 + u_{xy}^2)$ and $N^2(u) = \lambda^{-2} \mathcal{L}(u)$; hence (by (95)),

$$\tau_2 = 2\lambda^{-2} (u_{xx}^2 + u_{xy}^2) - \lambda^{-4} \mathcal{L}(u). \quad (101)$$

Then (by (100)-(101)) the Euclidean metric $dx^2 + dy^2$ is $\mathbb{R}\nabla u$ -critical if

$$-2(u_{xx}^2 + u_{xy}^2) + \lambda^{-2} \mathcal{L}(u) + \lambda^{-4} \mathcal{L}(u)^2 = 0. \quad (102)$$

If, for instance, $f(z) = z^2$ then $U = \mathbb{C} \setminus \{0\}$ and $\lambda = 2r$ with $r = (x^2 + y^2)^{\frac{1}{2}}$. Also $\mathcal{L}(u) = 8u$ and an inspection of (102) shows that Euclidean metric is not $\mathbb{R}\nabla u$ -critical.

The reader can find more examples for codimension one foliations in [2].

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